Classical Particle in a Box with Random Potential: exploiting rotational symmetry of replicated Hamiltonian.

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Abstract

We provide a detailed discussion of the replica approach to thermodynamics of a single classical particle placed in a random Gaussian $N\,(>> 1)-$dimensional potential inside a spherical box of a finite radius $L = R\sqrt{N}$. Earlier solutions of $R = \infty$ version of this model were based on applying the Gaussian Variational Ansatz (GVA) to the replicated partition function, and revealed a possibility of glassy phases at low temperatures. For a general R, we show how to utilize instead the underlying rotational symmetry and to arrive to a compact expression for the free energy in the limit $N \to \infty$ directly, without any need for intermediate variational approximations. This method reveals a striking similarity with the much-studied spherical model of spin glasses. Depending on the competition between the radius R and the curvature of the parabolic confining potential $\mu \geq 0$, as well as on the three types of disorder - short-ranged, long-ranged, and logarithmic - the phase diagram of the system in the (μ, T) plane undergoes considerable modifications. In the limit of infinite confinement radius our analysis confirms all previous results obtained by GVA. The paper has also a considerable pedagogical component by providing an extended presentation of technical details which are not always easy to find in the existing literature.

1 Introduction

In this paper we perform a detailed study of thermodynamics of a single classical particle confined to a spherical box filled in with an energy landscape described by a random Gaussian function $\mathcal{H}$ of N real variables $\mathbf{x} = (x_1, ..., x_N)$. Although the problem is meaningful for any N, we eventually will be mainly concerned with the limit of large $N \gg 1$ where we will be able to develop a systematic method of analysis. In fact, it is well known that such simple yet non-trivial models play a role of a laboratory for developing the methods allowing one to deal with problems of statistical mechanics where an interplay between thermal fluctuations and those due to quenched disorder is essential. The paradigmatic example of systems of this sort are spin glasses [1], but similar effects are frequently operational for polymers' behaviour in random environment, for phase separating interfaces in random field models or for elastic manifolds pinned by random impurities. In general,

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presence of quenched disorder leaves no choice but to employ the so-called replica trick, which is a heuristic way of extracting the averaged free energy of the system from moments of the partition function. The book [2] may serve as a modern introduction to this problematic.

Both conceptual and technical difficulties of dealing with statistical mechanics of disordered systems stem from the fact that many features of their low temperature dynamics and thermodynamics are dominated by presence of a huge number of metastable states (both minima and saddle points of various types) in the energy functional in configuration space. At finite temperature those features may generate a complicated free energy landscape, and that structure is responsible both for unusual equilibrium properties (an "ergodicity breaking", see [1, 2]) as well as for a complicated long-time dynamical behaviour. The latter manifests itself, in particular, through slow relaxation and aging effects [3]. At the level of static properties the broken ergodicity is reflected in an intricate pattern of spontaneous replica symmetry breaking discovered originally by Parisi [4] in the framework of the Sherrington-Kirkpatrick [5] model of spin glass with infinite-range interactions. Attempts to relate that picture to the properties of stationary points of the corresponding free energy landscape started long ago from the pioneering TAP paper[6] and is still a rather active field of research, see e.g. [7, 8] and references therein for a general account, and [9, 10, 11] for a treatment of the model of the present type.

In this broad context, the model of a single particle in a random potential has played an important role in physics of disordered system. It has enjoyed quite a long history of research starting from early works by Mezard and Parisi [12], and Engel [13] on static properties of such a system, followed by Franz and Mezard [14] and Cugliandolo and Le Doussal [3] papers on the corresponding dynamics. To have bona fide thermodynamics one has to ensure that the partition function Z of the model is well-defined for any realization of the random potential. This is usually achieved by introducing a sort of a confining potential $V_{con}(\mathbf{x})$ which prohibits escape of the particle to infinity. The standard choice is to use a parabolic potential uniform in all directions $V_{con}(\mathbf{x}) = \frac{\mu}{2}\mathbf{x}^2$. The curvature $\mu > 0$ then plays together with the temperature T a role of the main control parameter of the system, and one of standard goals of the theory is to investigate the phase diagram in the (μ, T) plane. Technically, the problem amounts to calculating the ensemble average of the equilibrium free energy $F = -T \ln Z$. The replica trick allows one to represent integer moments $\langle Z^n \rangle$ of the partition function in a form of some multivariable non-Gaussian integrals, and one then faces the usual problem of finding ways of evaluating those integrals for need of performing the replica limit $n \to 0$. The crucial step allowing one to achieve further progress was suggested by Mezard and Parisi [12] and is widely known as the Gaussian Variational Ansatz (GVA). Loosely speaking, it amounts to replacing the non-Gaussian integrands with trial Gaussian ones and employing the Feynman-Bogoliubov variational procedure to find best possible Gaussian approximation to the true free energy in the replica limit, see [12, 13] for a detailed discussion. Proceeding in this way, it turned out to be possible to employ again the Parisi scheme of spontaneous replica symmetry breaking at the level of the trial free energy, and to reveal the glassy nature of the

lower-temperature phase of the model. Since its induction GVA method became one of the most popular tools of dealing with quenched disorder. It allowed not only to get useful insights into static properties of glassy systems of quite a diverse nature (see e.g. [15, 16, 17, 18, 19]), but was eventually adopted to study the corresponding nonequilibrium dynamics as well [3, 20, 21]. Such a flexibility of GVA is based on the fact that it can be formally applied in any dimension N. The accumulated experience shows that the physical quantities calculated in such a way usually agree well with the results of numerical simulations on qualitative, and sometimes even quantitative level. As a result, GVA presently serves as a standard reference point for comparison with any new approach to the problem, as e.g. with a recently developed functional renormalization group method [22].

One should nevertheless recall that the exploitation of GVA lacks formal mathematical justification for finite dimensions. To this end, Mezard and Parisi provided an argument[12], see also [23], that being a variant of the Hartree-Fock approach the method should actually yield exact results in the limit $N \to \infty$. Another intrinsic feature of GVA is that the whole procedure is essentially based on a possibility to evaluate explicitly some intermediate Gaussian integrals. For the models defined in a restricted geometry presence of a geometric confinement may make applications of GVA less convenient. This is precisely the case for the class of models to be treated in the present paper: a particle confined to an impenetrable spherical box of some finite radius $L = R\sqrt{N},\ R < \infty$ filled with a Gaussian random potential. To make contact with previous works on the problem we retain also the parabolic confinement term, so that our model is characterised by both R and μ as control parameters. Our main technical observation is that the model is exactly solvable for any value of those parameters in the limit $N \to \infty$ without any need of introducing GVA. Rather, our method is based on observing that the replicated partition function possesses a high degree of invariance in the replica space: an arbitrary simultaneous $O(N)$ rotation of all n replica vectors $\mathbf{x}_a$. An efficient method of dealing with such integrals was developed recently in the work of one of the authors [24, 25] within the framework of the theory of random matrices. This method allows us to arrive to the effective free energy functional in the replica space in the most economic, and we believe elegant way. The subsequent analysis follows the standard route of using the Parisi Ansatz for the spontaneous replica symmetry breaking pattern, and the nature of the low-temperature phase is known to depend very much on the decay properties of the covariance function of the disorder potential[12, 13]. Actually, we propose a simple mathematical criterion which allows one to formally discriminate between the long-ranged and short-ranged disorder, and also suggests to single out potentials with logarithmically growing correlations as a separate intermediate class. For all types of the disorder we provide a detailed derivation of the free energy functional, the stability analysis, and a thorough description of the most important features of the resulting phase diagram. The latter undergoes considerable modifications reflecting a competition between the confinement provided by the radius R and the curvature of the parabolic potential μ. For the limiting case of infinite confinement radius our results faithfully reproduce all those fol-

lowing from earlier applications of GVA. This is just another explicit verification of the expected exactness of GVA in the limit $N \to \infty$.

One more point we find appropriate to mention here is that the Parisi scenario of replica symmetry breaking for spin glass models is changing presently its status from a powerful heuristic method of theoretical physics to an essentially rigorous mathematical procedure, well-controlled in the case of models of infinite range. This important change is mainly due to recent seminal results by Talagrand [26, 27], based on earlier works by Guerra [28], see also interesting works by Aizenman, Sims and Starr [29]. In particular, Talagrand was able to demonstrate that the equilibrium free energy emerging naturally in the Parisi scheme of replica symmetry breaking is indeed the correct thermodynamic limit of the free energy of the paradigmatic Sherrington-Kirkpatrick model and, more recently, for the so-called spherical model of spin glasses, originally studied by one of the present authors in [30]. It is natural to expect that similar justification should be possible also for other types of models with glassy thermodynamics, the model of a particle in random potential being the most natural candidate. However, the solution of the problem in the framework of GVA does not seem to be a promising starting point for such a verification. On the other hand, the expression for the free energy emerging in our approach is actually very close in its form to those emerging in the spherical model of spin glasses (see discussion in the end of the next section). The revealed similarities of our problem to the spherical spin glass model give strong evidence in favour of applicability of Talagrand's method for the model under consideration. We leave a detailed investigation of this issue for a future work.

Finally, we hope that our presentation has also a certain pedagogical value by providing extended description of a few technical details known to experts, but which are not always easy to find in the existing literature.

2 Definition of the model and its formal treatment by replica method.

As was discussed in the introduction, we consider a classical particle confined to an impenetrable spherical box of some finite size L. To ensure the non-trivial behaviour in the limit $N \to \infty$, one has to scale the radius of the sphere with N, and we denote the corresponding domain as $\{D_N : \mathbf{x}^2 \leq N\, R^2\}$ [2].

As usual, the main object of interest for us is to calculate the ensemble average of the free energy

$$F = -T\, \ln Z, \quad Z = \int_{D_N} \exp -\beta \mathcal{H}(\{\mathbf{x}\})\, d\mathbf{x}\,, \tag{1}$$

where $\beta = 1/T$ stands for the inverse temperature. The average of the logarithm over the disorder (denoted in the present paper by angular brackets) is performed with the help of

[2] For some applications, as e.g. [10] and [11], it is useful to keep in mind that the volume $V_N(R)$ of such a sphere in the large N limit behaves asymptotically as $V_N \approx L_e^N$ where $L_e = \sqrt{2\pi e}\ R$.

the standard replica trick, i.e. the formal identity

$$\langle \ln Z \rangle = \lim_{n\to 0} \frac{1}{n} \ln \langle Z^n \rangle, \quad Z^n = \int_{D_N} e^{-\beta \sum_{a=1}^n \mathcal{H}\{\mathbf{x_a}\}} \prod_{a=1}^{n} d\mathbf{x}_a \,. \tag{2}$$

The standard choice of the energy function for this problem is

$$\mathcal{H}\{\mathbf{x}\} = \frac{\mu}{\mathbf{2}} \sum_{\mathbf{k=1}}^{\mathbf{N}} \mathbf{x}_{\mathbf{k}}^{\mathbf{2}} + \mathbf{V}(\mathbf{x_1}, ..., \mathbf{x_N}) \tag{3}$$

with $\mu > 0$. A random Gaussian-distributed potential $V(\mathbf{x})$ is characterized by zero mean and the variance specified by the pair correlation function which we choose in the form ensuring stationarity:

$$\langle V(\mathbf{x}_1)\, V(\mathbf{x}_2) \rangle = N\, f\left(\frac{1}{2N}(\mathbf{x}_1 - \mathbf{x}_2)^2\right) . \tag{4}$$

The previous analysis [12, 13] revealed that one should essentially distinguish between two rather different situations. The first describes the case of a short-range correlated disorder corresponding to functions $f(x)$ vanishing at infinity, with $f(x) = e^{-x}$ being a typical representative of that class. In the second case the correlations are long ranged, and at large distances the potential grows in such a way that:

$$\left\langle [V(\mathbf{x}_1) - V(\mathbf{x}_2)]^2 \right\rangle \propto (\mathbf{x}_1 - \mathbf{x}_2)^{2\gamma} , \tag{5}$$

the exponent γ to be chosen in the range $0 < \gamma < 1$. The particular case $\gamma = 1/2$ corresponds to a potential $V(\mathbf{x})$ being the standard Brownian motion. Although formally $V(\mathbf{x})$ in the latter case can not satisfy the property Eq.(4) as its variance $\langle V^2(\mathbf{x}) \rangle$ is obviously position-dependent, one can easily satisfy oneself that such a difference is completely immaterial for the free energy calculations. As a result, we always assume for the long-ranged disorder the validity of Eq.(4) with the choice:

$$f(x) = f(0) - g^2 x^\gamma, \quad f(0) > 0, \quad 0 < \gamma < 1. \tag{6}$$

In what follows we will be able to formulate a certain criterium relating the nature of the low-temperature glassy phase of the model with the shape of the correlation functions $f(x)$, see Eq.(59) and the discussion around it. According to that criterium, in a broad class of random potentials with short-range correlations the glassy phase will be described by the so-called one-step replica symmetry breaking (1RSB), whereas all long-ranged potentials Eq(6) are characterized by the full replica symmetry breaking (FRSB). The criterium also suggests naturally to single-out as a special case the logarithmically growing correlations, that is

$$f(x) = -g^2 \ln(x + a^2), \quad a^2 < 1. \tag{7}$$

We shall see that such a choice leads to the phase diagram which combines some features typical for the short-ranged behaviour and others for the long-ranged types of disorder. In

this sense the logarithmically growing correlations should be considered as a marginal case intermediate between the two broad classes described above. It is interesting to mention that the choice of a glassy model with logarithmically growing correlations Eq.(7) is not purely academic, as such objects actually emerge e.g. in a context of statistics of the wave function in disordered two-dimensional systems [31].

Actually, a few initial steps in evaluation of the free energy are precisely the same for all types of disorder, provided the latter is of the Gaussian nature. Performing the averaging over the disorder in Eq.(2), we in the standard way arrive at the following expression:

$$\langle Z^n \rangle = e^{\frac{\beta^2}{2}Nnf(0)} \int_{D_N} \exp -\beta H_n\{\mathbf{x}_a\} \prod_{a=1}^{n} d\mathbf{x}_a \,, \tag{8}$$

where

$$H_n\{\mathbf{x_a}\} = \frac{\mu}{2}\sum_{a=1}^{n} \mathbf{x}_a^2 - N\beta \sum_{a<b} f\left(\frac{1}{2N}(\mathbf{x}_a - \mathbf{x}_b)^2\right) \,. \tag{9}$$

So far all our manupulations were exact. To achieve further progress one has to suggest an efficient way of working with the resulting multidimensional integral. In the standard model with infinite box radius $R = \infty$ Mezard and Parisi[12] suggested to deal with apparently non-Gaussian character of the integrand by replacing the exact replicated Hamiltonian $H_n\{\mathbf{x_a}\}$ with a trial Hamiltonian $H_n^{(t)}\{\mathbf{x_a}\}$ chosen to be Gaussian with respect to all variables $\mathbf{x_a}$, and then to apply a kind of variational principle to find the best possible Gaussian approximation.

Here we point out the possibility of a different route, valid for any value of the parameter R and requiring at this step no approximation. It is based on observing that the integrand in Eq.(8) in fact possesses a high degree of invariance: it depends on $N-$component vectors $\mathbf{x_a}$ only via $n(n+1)/2$ scalar products $q_{ab} = \mathbf{x_a}\mathbf{x_b},\ a \le b$, and is therefore invariant with respect to an arbitrary simultaneous $O(N)$ rotation of all vectors $\mathbf{x}_a$. Moreover, our choice of the integration domain respects this invariance.

An efficient method of dealing with such integrals is based on the possibility of rewriting the integral

$$J_{N,n} = \int_{|\mathbf{x}_1|<L} ... \int_{|\mathbf{x}_n|<L} \mathcal{I}_x(\mathbf{x}_1, ..., \mathbf{x}_n)\, d\mathbf{x}_1 \dots d\mathbf{x}_n \tag{10}$$

whose integrand $\mathcal{I}_x(\mathbf{x}_1, ..., \mathbf{x}_n)$ possesses such type of invariance in an alternative form as

$$J_{N,n} = \mathcal{C}_{N,n} \int_{D_N^{(Q)}} \mathcal{I}_Q(Q)\, \det Q^{(N-n-1)/2}\, dQ \,, \tag{11}$$

provided $N \ge n+1$. Here in Eq.(11) the original integrand $\mathcal{I}_x(\{\mathbf{x}_a\}))$ is expressed as a function $\mathcal{I}_Q(Q)$ of $n \times n$ real symmetric positive semidefinite matrix $Q \ge 0$ whose entries are precisely those q_{ab}, introduced above. The integration domain $D_N^{(Q)}$ is simply $D_N^{(Q)} = \{Q \ge 0,\, q_{aa} \le NR^2,\, a = 1, \dots n\}$, the volume element is $dQ = \prod_{a\le b} dq_{ab}$ and the

proportionality constant is given explicitly by

$$\mathcal{C}_{N,n} = \frac{\pi^{\frac{n}{2}\left(N-\frac{n-1}{2}\right)}}{\prod_{k=0}^{n-1}\Gamma\left(\frac{N-k}{2}\right)} \tag{12}$$

Although some use of similar formulae in statistical mechanics can be traced back to [32], the full potential of the transformation from Eq.(10) to Eq.(11) seems to be revealed in the work of one of the authors [24, 25] where it was rediscovered in the context of random matrix theory using Ingham-Siegel matrix integrals. Since then the formula found applications in physics of disordered systems, see e.g [33] for an elegant derivation and further use.

Applying this procedure to our case and using the subsequent rescaling $Q \to NQ$ yields the following exact expression for the averaged replicated partition function:

$$\langle Z^n \rangle = \mathcal{C}_{N,n} N^{Nn/2} e^{\frac{\beta^2}{2}Nnf(0)} \int_{D_Q} (\det Q)^{-(n+1)/2} \, e^{-\beta N \Phi_n(Q)} \, dQ \tag{13}$$

where

$$\Phi_n(Q) = \frac{\mu}{2}\sum_{a=1}^{n} q_{aa} - \frac{1}{2\beta}\ln(\det Q) - \beta\sum_{a<b} f\left[\frac{1}{2}(q_{aa}+q_{bb}) - q_{ab}\right] \tag{14}$$

and N is assumed to satisfy the constraint $N > n$. The final integration domain D_Q is already $N-$independent: $D_Q = \{Q \geq 0,\, q_{aa} \leq R^2,\, a = 1, \ldots n\}$.

The form of the integrand in Eq.(13) is precisely one required for the possibility of evaluating the replicated partition function in the limit $N \to \infty$ by the Laplace ("saddle-point") method[3]. Taking into account the expressions Eqs.(1), (2), and (12) the free energy of our model is then given by

$$F_\infty = \lim_{N\to\infty} \frac{1}{N}\langle F \rangle = -\frac{T}{2}\ln(2\pi e) - \frac{1}{2T} f(0) + \lim_{n\to 0}\frac{1}{n}\Phi_n(Q) \tag{15}$$

where the entries of the matrix Q are chosen to satisfy the conditions: $\frac{\partial \Phi_n(Q)}{\partial q_{ab}} = 0$ for $a \leq b$. This yields, in general, the system of $n(n+1)/2$ equations:

$$\mu - \frac{1}{\beta}\left[Q^{-1}\right]_{aa} - \beta \sum_{b(\neq a)}^{n} f'\left[\frac{1}{2}(q_{aa}+q_{bb}) - q_{ab}\right] = 0, \quad a = 1, 2, \ldots, n \tag{16}$$

[3] One should note that as long as one is interested only in finding the leading exponential factors in $N \to \infty$ limit, one can in principle follow a different route. Namely, impose $n(n+1)/2$ constraints $Q_{ab} = \frac{1}{N}\mathbf{x_a}\mathbf{x_b}$ through the integral Fourier representations involving $n(n+1)/2$ auxiliary fields λ_{ab}, and take the saddle-point both in $\lambda-$ and $Q-$ variables. The same asymptotic result Eq.(15) then follows after simple manipulations. We however believe that the use of the elegant mathematical procedure based on Eq.(11) is more conceptually clear, and appealing aesthetically. We hope that it may also provide a useful basis for calculating $1/N$ corrections, and perhaps for a rigorous mathematical treatment of the problem.

and

$$-\frac{1}{\beta}\left[Q^{-1}\right]_{ab}+\beta f'\left[\frac{1}{2}(q_{aa}+q_{bb})-q_{ab}\right]=0, \quad a\neq b \tag{17}$$

where $f'(x)$ stands for the derivative df/dx.

One should also ensure that the solutions to these equations respects the constraint $q_{aa}\leq R^2$ for all $a=1,\ldots,n$ imposed by presence of the boundaries of the integration domain D_Q. We will shortly see that in the replica limit $n\to 0$ that condition will be violated for some regions of parameters T,μ. If this happens, Eqs.(16) should be simply replaced by equalities $q_{aa}=R^2$, and the solution to the remaining set Eq.(17) should be sought with those constraints imposed. One then notices that for $R=1$ the resulting expression for the free energy formally coincides , up to a trivial constant term, with that obtained for the so-called "spherical" mean field model of spin glasses. Various features of the latter model attracted a lot of research interest in recent years, see e.g. [30, 34, 35, 27].

The correspondence between the free energies of the two models which is so apparent in our approach deserves a short comment. In one of its recent incarnations the Hamiltonian $\mathcal{H}(\sigma)$ of the spherical model was defined as [27]

$$\mathcal{H}(\sigma)=\sum_{p\geq 1}\frac{J_p}{N^{(p-1)/2}}\sum_{i_1,\ldots,i_p} g_{i_1,\ldots,i_p}\sigma_{i_1}\ldots\sigma_{i_p} \tag{18}$$

in terms of $N-$component vectors σ spanning the sphere of radius $\sqrt{N}$. Here $g_{i_1,\ldots,i_p}$ denote independent Gaussian variables with mean zero and unit variance. The original version [30] of the same model can be shown to yield precisely the same free energy in the thermodynamic limit[27]. Immediate consequence of the definition Eq.(18) is that the correlation function $\frac{1}{N}\langle\mathcal{H}(\boldsymbol{\sigma_1})\mathcal{H}(\boldsymbol{\sigma_2})\rangle_g$ depends on the vectors σ_1 and σ_2 only via the scalar product $(\sigma_1\sigma_2)$. This is precisely the property sufficient to ensure global $O(N)$ invariance of the replicated partition function of the model, and our method of deriving the mean free energy goes through without any modifications, and may well be the shortest possible. The spherical constraint $\sum_{i\leq N}\sigma_i^2=N$ simply translates into the condition $R=1$.

3 Analysis of the phase diagram of the model within the replica symmetric ansatz.

Our procedure of investigating the equations Eqs.(16,17) in the replica limit $n\to 0$ will follow the standard pattern suggested by developments in spin glass theory. We first seek for the so-called "replica symmetric" solution, and then investigate its stability in the (μ,T) plane. When the replica symmetric solution is found inadequate, it will be replaced by the hierarchical ("Parisi", or "ultrametric") ansatz for the matrix elements q_{ab}, with various levels of replica symmetry breaking. Since the full analysis contains a lot of features to be explained in detail, the reader may wish to cast a regular look at the resulting phase diagrams fig1, fig.2 and fig.3 in the process of reading.

The Replica Symmetric Ansatz amounts to searching for a solution to Eqs.(16,17) within subspace of matrices Q such that $q_{aa} = q_d$, for any $a = 1, \ldots n$, and $q_{a<b} = q_0$, subject to the constraints $0 < q_0 \le q_d \le R^2$. Inverting such a matrix Q yields again the matrix of the same structure, with the diagonal entries all given by

$$p_d = \frac{q_d + q_0(n-2)}{(q_d - q_0)(q_d + q_0(n-1))} \tag{19}$$

and off-diagonal entries given by

$$p_0 = -\frac{q_0}{(q_d - q_0)(q_d + q_0(n-1))} \tag{20}$$

Note, that

$$p_d - p_0 = \frac{1}{q_d - q_0} \tag{21}$$

The system Eqs.(16,17) is reduced in this way to two equations, which we write directly in the replica limit $n \to 0$ as

$$\mu - T\,p_d + \frac{1}{T} f'\left(q_d - q_0\right) = 0 \tag{22}$$

$$-\,T\,p_0 + \frac{1}{T} f'\left(q_d - q_0\right) = 0 \tag{23}$$

This system of equations is easy to solve employing the relation (21), and to obtain

$$q_d = \frac{T}{\mu} - \frac{1}{\mu^2} f'\left(\frac{T}{\mu}\right), \quad q_0 = -\frac{1}{\mu^2} f'\left(\frac{T}{\mu}\right) \tag{24}$$

In order this solution to be sensible one first of all has to require $f'(x) < 0$, which we always assume to hold in our model. In addition, the solution Eq.(24) can hold only as long as $q_d \le R^2$, which in view of the above expressions amounts to the condition

$$R^2 - \frac{T}{\mu} + \frac{1}{\mu^2} f'\left(\frac{T}{\mu}\right) \ge 0\ . \tag{25}$$

When the above inequality is violated, we should rather use $q_d = R^2$, and find q_0 from the "spherical model" type equation

$$\frac{q_0}{(R^2 - q_0)^2} + \frac{1}{T^2} f'(R^2 - q_0) = 0 \tag{26}$$

following immediately from Eqs.(23,20).

Let us briefly discuss general properties of the boundary line $T_b(\mu)$ separating the "$\mu-$dominated" regime from the "R-dominated" one in (μ, T) plane, for a fixed value of

the confining radius R. For a given value of μ the value of T_b is obtained by solving the equation

$$R^2 - \frac{T_b}{\mu} = -\frac{1}{\mu^2} f'\left(\frac{T_b}{\mu}\right) . \tag{27}$$

When analysing this equation we shall assume in addition to the condition $f'(x) < 0$ two more conditions: the uniform concavity condition $f''(x) > 0$ as well as the condition $f'''(x) < 0$. Sensibility of that choice will be justified by analysis of patterns of spontaneous replica symmetry breaking in subsequent sections, see discussions around Eq.(59). We also assume that $f'(x) \to 0$ for large x, as is indeed the case for all types of disorder in the original model.

For the short-range disorder the values $f'(0) < \infty$ and $f''(0) < \infty$. It is convenient to define for subsequent use two quantities

$$\mu_c = \frac{1}{R}\sqrt{-f'(0)}, \quad R_{cr} = \sqrt{-\frac{f'(0)}{f''(0)}} \tag{28}$$

Then a simple graphical analysis of Eq.(27) shows that for $\mu > \mu_c$ that equation has a single solution $T_b(\mu)$ tending asymptotically to $T_b = \mu R^2$ for $\mu \gg \mu_c$. In contrast, for $\mu < \mu_c$ the number of solutions essentially depends on the value of the confining radius R. Defining R_{cr} as in Eq.(28), we find that for $R < R_{cr}$ the condition $\mu < \mu_c$ implies that no such solution T_b exists at all. This fact corresponds to the picture of monotonically increasing curve $T_b(\mu)$ starting from the point $(\mu_c, 0)$ in the (μ, T) plane. In the opposite case $R > R_{cr}$ there are two solutions $T_{b1} < T_{b2}$ in the whole interval $\mu_0 < \mu \leq \mu_c$, with $T_{b2} - T_{b1} \to 0$ as $\mu \to \mu_0$. The value of μ_0 and the corresponding temperature value $T_0 = T_{b1} = T_{b_2}$ can be found as

$$\mu_0 = \sqrt{f''(\tau_0)}, \quad T_0 = \mu_0 \tau_0, \tag{29}$$

with τ_0 being the solution of the equation

$$R^2 = h(\tau_0), \quad h(\tau) = \tau - \frac{f'(\tau)}{f''(\tau)}. \tag{30}$$

Note, that $dh/d\tau = f'(\tau)f'''(\tau)/[f''(\tau_0))]^2 > 0$ according to our assumptions. Thus, the right-hand side of Eq.(30) is a monotonously increasing function, and thus the equation has a (unique) solution $\tau_0 \geq 0$ as long as $R^2 \geq h(0) = R_{cr}^2$. In contrast, for $R < R_{cr}$ the equation Eq.(30) has no solutions.

Finally, for $\mu < \mu_0$ the equation (27) has no more solutions.

Relegating similar analysis of the long-ranged as well as the logarithmic correlations to the end of the section, we now discuss the last important ingredient of the procedure: the issue of the stability of the emerging solution against fluctuations in the replica space. Indeed, the very essence of the saddle-point method calls for a check of the replica symmetric solution being locally stable, in the sense of corresponding to the true extremum

(in the replica limit $n \to 0$, to a maximum) of the functional $\Phi(Q)$. This should be judged by analysing the eigenvalues of the matrix $G_{ab,cd} = \frac{\partial^2 \Phi}{\partial q_{ab} \partial q_{cd}}$ at this solution. Such analysis along the lines of the classical De-Almeida and Thouless (AT) paper[36] is presented in the Appendix B. The main outcome is that the μ-dominated replica symmetric solution Eq.(24) is locally stable as long as

$$\mu^2 - f''\left(\frac{T}{\mu}\right) \geq 0, \tag{31}$$

whereas the "spherical model"-type solution satisfying Eq.(26) is stable provided

$$\frac{1}{(R^2 - q_0)^2} - \frac{1}{T^2} f''(R^2 - q_0) \geq 0. \tag{32}$$

Combining the latter with Eq.(26) for q_0 one easily finds that the domain of the replica-symmetric stability in the R-dominated regime is given by

$$T \geq \tau_0 \sqrt{f''(\tau_0)} \tag{33}$$

where τ_0 is precisely the solution of Eq.(30).

We will see shortly that both conditions Eq.(31) and Eq(33) could be violated for low enough values of μ and T below the so-called de-Almeida-Thouless line $T_{AT}(\mu)$. Actually, in the $R-$dominated region of the phase diagram such a line is parallel to the $\mu-$axis, as the solution τ_0 of Eq.(30) determining the right-hand side of (33) depends only on R and hence is $\mu-$ independent. In particular, for the limiting case $\mu = 0$ the expression for the whole de-Almeida-Thouless line in (R, T) plane is just $T_{AT}(R) = \tau_0 \sqrt{f''(\tau_0)}$ (see e.g fig.4(c) for a particular choice of the short-ranged potential).

Let us see how these features are incorporated into the analysis of the boundary $T_b(\mu)$ between the μ-dominated and the $R-$dominated regions performed by us above. In the case of a short-range disorder with $f''(0) < \infty$ studied above we can easily see that the de-Almeida Thouless line $T_{AT}(\mu)$ given by the equality sign in Eq.(31) must end up for zero temperature at the point $\mu_{AT} = \sqrt{f''(0)}$. As a simple consequence, for small confining radius $R < R_{cr}$, with R_{cr} again given by Eq.(28), the whole instability region falls outside the domain of validity of μ-dominated solution. To find whether the replica-symmetric solution is stable one therefore has to use instead the equation given by the equality sign in Eq.(32). As the latter equation actually does not have at all roots for $R < R_{cr}$, we conclude that for the present model the replica symmetric solution is always stable for such values of the confining radius R. In short, the small box size $R < R_{cr}$ precludes possibility of the replica symmetry breaking, hence the glassy behaviour of the models with short-ranged correlations, see fig.3a[4]

[4]Note that in recent works [10, 11] precisely the same lengthscale R_{cr} appeared in the analysis of the "geometric complexity" associated with the zero-temperature limit of the present model. Namely, the samples with $R < R_{cr}$ were found to be unable to support the existence of exponentially many saddle points in their energy landscape. Such an existence seems to be a necessary feature of a phase with glassy behaviour[11].

The situation changes crucially for $R > R_{cr}$. First of all, now the lines $T_b(\mu)$ and the AT line $T_{AT}(\mu)$ can indeed intersect each other in (T,μ) plane. Curiously enough, the point of intersection turns out to coincide with the point (μ_0, T_0), see Eq.(29), which emerged in the previous analysis as the "leftmost" point on the curve $T_b(\mu)$, such that for $\mu < \mu_0$ the equation (27) has no solutions. For $\mu > \mu_0$ the branch T_{b1} drops therefore below the AT line and hence has no meaning. At the same time for another branch $T_{b2} > T_{AT}$ and hence that solution survives. We shall see in subsequent sections that our analysis of the phase with broken replica symmetry will provide us with a modified expression for the boundary line $T_{b1}(\mu)$ extending from the point (μ_0, T_0) down to zero temperature.

For $\mu < \mu_0$ we are in the domain of validity of the R-dominated solution, and the corresponding AT temperature should be given by $\mu-$ independent value T_{AT} from Eq.(33). The latter expression now makes full sense as the corresponding solution τ_0 indeed exists. Finally, in view of Eq.(29) it is evident that the $\mu-$ independent value of T_{AT} is simply T_0 everywhere in R-dominated phase, so that the R-dominated and $\mu-$ dominated AT lines indeed meet each other at the same point (μ_0, T_0) of the phase diagram, as was natural to expect.

So far our analysis assumed the case of a short-range disorder. For the case of long-ranged disorder, Eq.(6), the overall structure of the line $T_b(\mu)$ is in many respects similar, but has some peculiarities. In particular, there will be no analogue of the critical value of the confinement radius $R = R_{cr}$ in this case. Below we proceed in presenting a brief analysis of the situation for any value of the exponent $\gamma \in (0,1)$. Using $f'(x) = -\gamma g^2 x^{\gamma-1}$, the equation Eq.(25) for the boundary line $T_b(\mu)$ between the $\mu-$dominated and spherical model like regions in the (μ, T) diagram can be written as

$$F(u) = u^{\frac{2-\gamma}{1-\gamma}} - R^2 u + \gamma \frac{g^2}{\mu^2} = 0, \quad \text{with } u = \left(\frac{T_b}{\mu}\right)^{1-\gamma} \tag{34}$$

We see that the function $F(u)$ has its single minimum for $u > 0$ at $u = u_{min} = \left(\frac{1-\gamma}{2-\gamma} R^2\right)^{1-\gamma}$, and $F(0) > 0$, $F(u \to \infty) > 0$. Hence, the equation $F(u) = 0$ has two positive solutions $u_{1,2}$ as long as $F(u_{min}) < 0$, and no solutions if $F(u_{min}) > 0$. Then a simple calculation shows that two branches $T_{b1} < T_{b2}$ exist as long as

$$\mu > \mu_0 = \frac{g\gamma^{1/2}(2-\gamma)^{1-\gamma/2}}{R^{2-\gamma}(1-\gamma)^{\frac{1-\gamma}{2}}}\,. \tag{35}$$

At $\mu = \mu_0$ those two branches merge: $T_{b1} = T_{b_2} = T_0$ which implies $u_2 = u_1 = u_{min}$. The corresponding characteristic temperature can be found as

$$T_0 = \mu_0 u_{min}^{\frac{1}{1-\gamma}} = \frac{g\gamma^{1/2}(2-\gamma)^{-\gamma/2} R^\gamma}{(1-\gamma)^{-\frac{1+\gamma}{2}}}\,. \tag{36}$$

Finally, we should take the presence of the de Almeida-Thouless conditions Eqs.(31,33) into consideration. Recall that for the case of a long-ranged disorder $f''(0) = \infty$ in

contrast to the finite value typical for the short-range case. As a consequence, the AT line now extends in the (μ, T) plane to arbitrary large values of μ and is explicitly given by the equation

$$T_{AT} = \frac{\mu^{\frac{\gamma}{\gamma-2}}}{[g^2\gamma(1-\gamma)]^{\frac{1}{\gamma-2}}} \,. \tag{37}$$

It is now easy to verify that the boundary line $T_b(\mu)$ which is given for $\mu \geq \mu_0$ by the two branches T_{b1} and T_{b2} intersects AT line precisely at the point (μ_0, T_0), with $u_{AT} = (T_{AT}/\mu)^{1-\gamma}$. Moreover, one can check that $F(u_{AT}) < 0$ for $\mu > \mu_0$. This simply implies that $T_{b1} < T_{AT} < T_{b2}$ for any $\mu > \mu_0$. Hence, the lower branch T_{b1} should be discarded as falling outside the domain of validity of the replica symmetric solution. The analysis of the phase with broken replica symmetry presented in the subsequent sections of this paper reveals that the line $T_{b1}(\mu)$ shall be rather replaced by a vertical line $\mu = \mu_0$ extending from the point (μ_0, T_0) down to zero temperature. Finally, for all values $\mu < \mu_0$ the de-Almeida - Thouless line $T_{AT}(\mu)$ is given by the $\mu-$independent temperature $T_{AT} = T_0(R)$. In particular, this expression just provides the de-Almeida-Thouless line in (R, T) plane for the limiting case $\mu = 0$.

At last, we provide the analysis of the replica symmetric solution for the logarithmic correlations, Eq.(7). In this case both AT line and the boundaries T_{b1}, T_{b_2} can be easily found explicitly in the whole $\mu-$dominated regime. They are given by

$$T_{AT}(\mu) = g - \mu\, a^2, \quad T_b(\mu) = \frac{\mu}{2}\left[R^2 - a^2 \pm \sqrt{(R^2+a^2)^2 - 4g^2/\mu^2}\right] , \tag{38}$$

where the first formula holds for $\mu \leq \mu_{AT} = g/a^2$, and in the second formula the upper sign corresponds to T_{b2}, and the lower to T_{b1}. The two branches of $T_b(\mu)$ meet at $\mu = \mu_0 = 2g/(R^2 + a^2)$ so that there is no T_b solution for $\mu < \mu_0$. The de-Almeida-Thouless line meets the boundary $T_b(\mu)$ at precisely μ_0. In order to make such intersection happen one has to ensure that $\mu_0 < \mu_{AT}$, which is possible only if the confinement radius R exceeds the critical value $R_{cr} = a$. As long as $R > R_{cr}$ for $\mu > \mu_0$ one can see that $T_{b1}(\mu) < T_{AT}(\mu) < T_{b2}(\mu)$, so that only the upper branch T_{b2} makes actually sense. A subsequent analysis of the phase with broken replica symmetry will again reveal that the line $T_{b1}(\mu)$ should be replaced by the vertical line $\mu = \mu_0$ everywhere in the glassy phase. Finally, for $\mu < \mu_0$ the de-Almeida-Thouless temperature is given by the $\mu-$ independent value $T_{AT} = T_0 = g\frac{R^2-a^2}{R^2+a^2}$. Again, for the limiting case $\mu = 0$ this expression provides the de-Almeida-Thouless line in the whole (R, T) plane. In particular, for $R \to R_{cr} = a$ we have $T_0 \to 0$, showing that there is no place for broken replica symmetry in the logarithmic case as long as $R < R_{cr}$.

To summarize our findings, we present in fig.1, fig.2 and fig.3 the resulting phase diagrams in (μ, T) plane for the particular choice (i) $f(x) = f(0) - g^2 x^{1/2}$ of long-range disorder, (ii) for the logarithmic correlations, and finally (iii) $f(x) = e^{-x}$ corresponding to the short range disorder. In all these cases the AT line can be found explicitly. For the $\mu-$dominated regime in case (i) $T_{AT} = \frac{g^{4/3}}{\mu^{1/3}2^{4/3}}$ and in case (iii) $T_{AT} = -2\mu \ln \mu$.

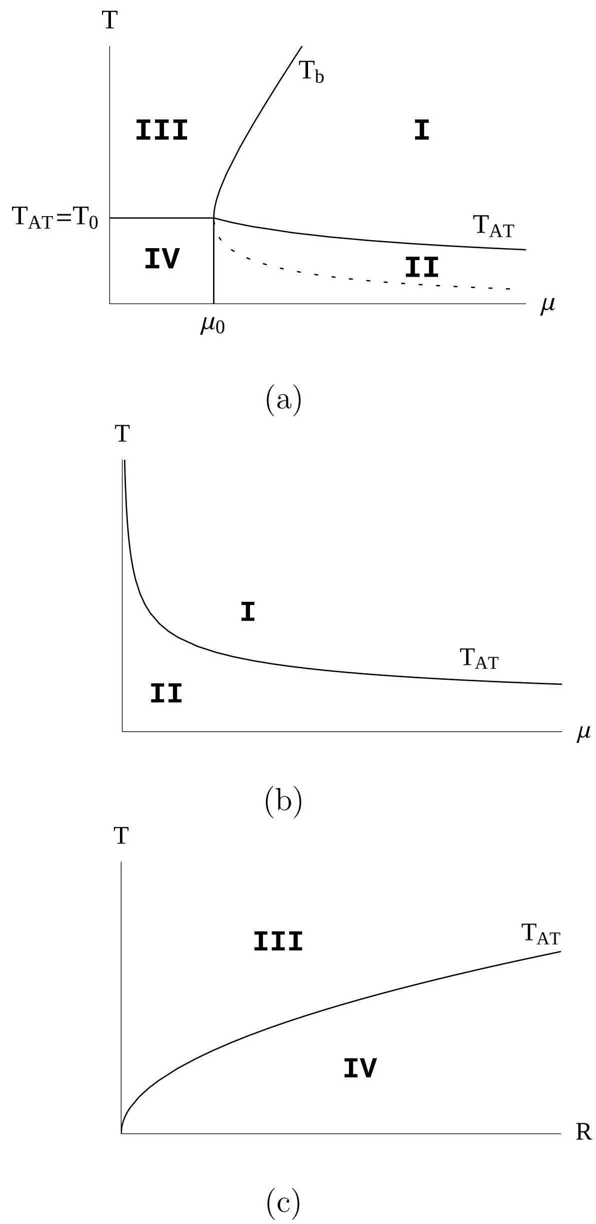

Figure 1: The phase diagrams for the long-ranged potential with $\gamma = 1/2$ and $g = 2$. The case (a) corresponds to $R = \sqrt{3}$, the case (b) to $R = \infty$, and the case (c) to the choice $\mu = 0$. Dotted line in the case (a) represents the wrong branch of the boundary T_b between the $R-$ dominated and $\mu-$ dominated phases with broken replica symmetry, and is replaced by the vertical full line. The notation for phases are as follows: **I** stands for $\mu-$dominated replica-symmetric (RS) phase; **II** for $\mu-$dominated glassy phase with broken RS; **III** for $R-$dominated RS phase, and **IV** for $R-$dominated glassy phase with broken RS;

The corresponding coordinates of the intersection point (μ_0, T_0) for the two models: (i) $\mu_0 = \frac{3^{3/4} g}{2 R^{3/2}}$, $T_0 = \frac{g R^{1/2}}{3^{1/4} 2}$ and (iii) $\mu_0 = e^{-(R^2-1)/2}$, $T_0 = (R^2-1)\, e^{-(R^2-1)/2}$. Note, that the branch T_{b1} of the solution to (27) has no particular meaning as it drops as a whole below the AT line. In the presented phase diagrams we also included the replacement of that line by the correct boundaries $T_b(\mu)$ everywhere in the glassy region $T < T_{AT}$.

We also presented for completeness the typical phase diagrams for systems with long-range and logarithmic correlations in two limiting cases: in the (μ, T) plane for $R = \infty$ and in (R, T) plane for $\mu = 0$. The de-Almeida-Thouless temeperature in the latter case is equal to $T_{AT} = T_0(R) = \frac{g R^{1/2}}{3^{1/4} 2}$ for the long-ranged potential with $\gamma = 1/2$. The corresponding diagram for the case of short-range correlations will be presented in the end of section Sec.(4.2).

We finish this section with writing down explicit expressions for the equilibrium free energy in the replica-symmetric solution. They are given correspondingly by

$$F_\infty = -\frac{T}{2} \ln\left(2\pi T/\mu\right) - \frac{1}{2T}\left[f(0) - f(T/\mu)\right] \tag{39}$$

for the $\mu-$ dominated region; in particular, $F_\infty|_{T=0} = f'(0)/2\mu$. As for the $R-$ dominated RS region, the free energy is given by

$$F_\infty = \frac{1}{2} R^2 (\mu - \frac{1}{t}) - \frac{T}{2} \ln\left(2\pi t T\right) - \frac{1}{2T}\left[f(0) - f(tT)\right]\;, \tag{40}$$

with t being the solution of the equation (see Eq.(26))

$$\frac{R^2}{t^2} - \frac{T}{t} + f'(tT) = 0 \tag{41}$$

In particular, $t|_{T=0} = R/\sqrt{-f'(0)}$ and $F_\infty|_{T=0} = \frac{1}{2}\mu\, R^2 - R\sqrt{-f'(0)}$. Along the line $T_b(\mu)$ given by Eq.(27) we obviously have $t = 1/\mu$ and the two expressions for the free energy indeed coincide. In particular, the two expressions for $F_\infty|_{T=0}$ are indeed equal for $\mu = \mu_c$, see Eq.(28).

4 Free energy functional within the Parisi scheme of replica symmetry breaking.

In the region below the AT lines one must discard the unstable replica symmetric solution in favour of one with the broken symmetry. In the present section we derive the expression for the free energy of our model in the phase with broken replica symmetry. We will follow a particular heuristic scheme of the replica symmetry breaking proposed originally by Parisi [4] in the framework of the SK model and employed for the case of the spherical spin glass in [30].

In our dealing with the free energy functional Eq.(14) we follow closely the method suggested in the paper by Crisanti and Sommers [30]. To make the present paper self-contained we choose to describe the procedure in Appendix A and to provide a few technical details skipped in [30]. The analysis is based on explicit calculation of the eigenvalues of the Parisi matrix Q.

We are actually interested in the replica limit $n \to 0$. According to the Parisi prescription explained in detail in the Appendix A, in such a limit the system is characterized by a non-decreasing function of the variable q denoted as $x(q)$. Such a function depends non-trivially on its argument in the interval $q_0 \leq q \leq q_k$, with $q_0 \geq 0$ and $q_k \leq q_d$. Outside that interval the function stays constant:

$$x(q < q_0) = 0, \quad \text{and} \quad x(q > q_k) = 1. \tag{42}$$

In general, the function $x(q)$ also depends on the increasing sequence of k positive parameters m_i satisfying the following inequalities

$$0 \leq m_1 \leq m_2 \leq \ldots \leq m_k \leq m_{k+1} = 1\,. \tag{43}$$

As shown in the Appendix A, in the replica limit the following identity must hold for any differentiable function $g(x)$:

$$\lim_{n\to 0} \frac{1}{n} Tr\left[g(Q)\right] = g\left(q_d - q_k\right) + \int_0^{q_k} g'\left(\int_q^{q_d} x(\tilde{q})\, d\tilde{q}\right)\, dq\,. \tag{44}$$

In particular, for the first two terms entering the replica free energy functional Eq.(14) application of the rule Eq.(44) gives

$$\lim_{n\to 0} \frac{1}{n}\left[\frac{\mu}{2} Tr\, Q - \frac{1}{2\beta} Tr \ln\left(Q\right)\right] = \frac{\mu}{2}\, q_d - \frac{1}{2\beta} \ln\left(q_d - q_k\right) - \frac{1}{2\beta}\int_0^{q_k} \frac{1}{\int_q^{q_d} x(\tilde{q})\, d\tilde{q}}\, dq\,. \tag{45}$$

The last term in Eq.(14) is also easily dealt with in the Parisi scheme (see Appendix A), where it can be written as

$$\lim_{n\to 0} -\frac{1}{n}\sum_{a\neq b} f\left[\frac{1}{2}(q_{aa}+q_{bb}) - q_{ab}\right] = \lim_{n\to 0}\sum_{l=0}^{k}(m_{l+1}-m_l) f(q_d - q_l) = \int_0^{q_d} f(q_d - q) x'(q)\, dq, \tag{46}$$

by using explicitly the derivative of the generalized function Eq. (109). Using integration by parts and taking into account the properties Eq.(42) we finally arrive at the required free energy functional for the phase with broken replica symmetry

$$\begin{aligned} F_\infty = {} & \frac{\mu}{2}\, q_d - \frac{T}{2}\left[\ln\left(2\pi e (q_d - q_k)\right) + \int_0^{q_k} \frac{1}{q_d - q_k + \int_q^{q_k} x(\tilde{q})\, d\tilde{q}}\, dq\right] \\ & + \frac{1}{2T}\left(f(q_d - q_k) - f(0) + \int_{q_0}^{q_k} f'(q_d - q) x(q)\, dq.\right) \end{aligned} \tag{47}$$

This functional should be now extremised with respect to the non-negative non-decreasing continuous function $x(q)$. In the $\mu-$dominated regime it also should be extremised with respect to q_d, whereas for the $R-$dominated situation the latter variable is fixed to be $q_d = R^2$.

4.1 Solution within the scheme of fully broken replica symmetry.

The actual analysis essentially depends on whether we consider the number of steps k in the Parisi scheme of the replica symmetry breaking (see Appendix A) to be finite, or allow it to be infinite. In the latter situation one conventionally speaks about a case of full replica symmetry breaking (FRSB), and in the rest of the present section we restrict our consideration to this specific case. The number of steps k is assumed to tend to infinity in such a way, that the sequence of the parameters m_l is replaced with a continuous variable $u \in [0,1]$, such that $m_{l+1} - m_l = \Delta m_l \to du$. The sequence of parameters q_i satisfying Eq.(105) is simultaneously transformed to a non-decreasing differentiable function $q(u)$ changing between its minimal value $q(u=0) = q_0$ and its maximal value $q(u=1) = q_k$. The function $x(q)$ defined in Eq.(109) is then assumed to be transformed to a non-decreasing function continuous in the interval $q_0 \le q \le q_k$, and at least once differentiable there. Outside that interval it satisfies Eq.(42). As is easy to see, the identity $x[q(u)] = u$ holds as long as $q_0 \le q(u) \le q_k$.

Variation of the free energy functional Eq.(47) with respect to such a function $x(q)$ as well as with respect to q_d yields

$$\delta F_\infty = \frac{1}{2}\left[\int_{q_0}^{q_k} \delta x(q) S(q) dq + \delta q_d\, P\right] = 0\,, \tag{48}$$

where

$$S(q) = T\int_0^q \frac{d\tilde{q}}{\left[q_d - q_k + \int_{\tilde{q}}^{q_k} x(q)\,dq\right]^2} + \frac{1}{T} f'(q_d - q) \tag{49}$$

and

$$P = \mu - \frac{T}{q_d - q_k} + T\int_0^{q_k} \frac{d\tilde{q}}{[q_d - q_k + \int_{\tilde{q}}^{q_k} x(q)\,dq]^2} + \frac{1}{T} f'(q_d - q_k) + \frac{1}{T}\int_{q_0}^{q_k} x(q) f''(q_d - q) dq\,. \tag{50}$$

Stationarity therefore always amounts to the condition

$$S(q) = 0,\ \forall q \in [q_0, q_k]\ , \tag{51}$$

and for the $\mu-$ dominated regime we must add also the equation $P = 0$. As $S(q) = 0$ implies $\frac{d}{dq} S(q) = 0$ we can differentiate Eq.(49) once, and get after a simple algebra the equation

$$q_d - q_k + \int_q^{q_k} x(\tilde{q})\, d\tilde{q} = \frac{T}{\sqrt{f''(q_d - q)}}, \quad \forall q \in [q_0, q_k]\ . \tag{52}$$

Next differentiation immediately yields the explicit formula for the function $x(q)$

$$x(q) = -\frac{T}{2}\frac{f'''(q_d - q)}{[f''(q_d - q)]^{3/2}}, \quad \forall q \in [q_0, q_k]\ . \tag{53}$$

What remains to be determined are the values for parameters q_0, q_k, q_d, the last one only for the $\mu-$ dominated case. To this end, we substitute the value $q = q_k$ into the relation Eq.(52) and obtain a simple equation for the difference $q_d - q_k$:

$$q_d - q_k = \frac{T}{\sqrt{f''(q_d - q_k)}} \ . \tag{54}$$

Next, we use the condition $S(q_0) = 0$. Exploitation of Eqs.(49,52) yields after a simple algebra one more relation:

$$q_0 = -\frac{f'(q_d - q_0)}{f''(q_d - q_0)} \ . \tag{55}$$

Finally, we can use already obtained relations to simplify drastically a somewhat cumbersome equation $P = 0$, see Eq.(50), which holds only in the μ-dominated regime. In particular, the explicit form Eq.(53) for the function $x(q)$ allows one to write:

$$\frac{1}{T}\int_{q_0}^{q_k} x(q) f''(q_d - q)dq = -\frac{1}{2}\int_{q_0}^{q_k} \frac{f'''(q_d - q)}{[f''(q_d - q)]^{1/2}} dq = [f''(q_d - q_k)]^{1/2} - [f''(q_d - q_0)]^{1/2}$$

Now taking into account the condition $S(q_k) = 0$ and the relation Eq.(54) allows one to see that the stationarity condition $P = 0$ amounts simply to

$$\mu^2 = f''(q_d - q_0) \ . \tag{56}$$

which in turn allows to rewrite Eq.(55) in the $\mu-$dominated regime as

$$q_0 = -\frac{1}{\mu^2} f'(q_d - q_0) \ . \tag{57}$$

The system of three equations Eq.(54), Eq.(56) and Eq.(57) for the $\mu-$ dominated regime or of the two equations Eq.(54), Eq.(55) for the $R-$ dominated regime allows one to determine the parameters q_0, q_k (and, if necessary, q_d) as long as the function $f(x)$ is specified. The corresponding equilibrium free energy can be written, after some algebraic manipulations exploiting Eq.(52,53,54,55), in the form valid for both regimes:

$$F_\infty = \frac{\mu}{2} q_d - \frac{T}{2} \ln\left[2\pi e (q_d - q_k)\right] + \frac{1}{2T}\left[f(q_d - q_k) - f(0) - (q_d - q_k) f'(q_d - q_k)\right]$$
$$-q_0 \sqrt{f''(q_d - q_0)} - \int_{q_0}^{q_k} \sqrt{f''(q_d - q)}\, dq, \tag{58}$$

where for the $R-$dominated phase we assume that $q_d = R^2$.

This seemingly completes our solution.

Consistency of the Parisi FRSB scheme requires, however, that the emerging function $x(q)$ given by Eq.(53) must be a real-valued non-negative non-decreasing function of its argument. Non-negativity and reality of this function is indeed ensured by earlier imposed conditions $f''(x) > 0$ and $f'''(x) < 0$. Non-negativity of the derivative of the right-hand

side in Eq.(53) yields as the necessary consistency condition a new inequality $A(q_d - q) \le 0$, where the function $A(x)$ is expressed in terms of $f(x)$ as

$$A(x) = \frac{3}{2}\left[f'''(x)\right]^2 - f''(x)f''''(x). \qquad (59)$$

Checking our three main choices for the function $f(x)$ - the long-ranged, the short ranged and the logarithmic - we observe that the above inequality is indeed strictly satisfied for any long-ranged potential Eq.(6) where

$$A(x) = \frac{1}{2}g^2\gamma^2(\gamma-1)^2(\gamma-2)x^{2(\gamma-3)} < 0, \quad \forall\gamma \in (0,1)\,.$$

The inequality is only marginally satisfied in the logarithmic case Eq.(7) where it is easy to see that $A(x) \equiv 0$. And it is strictly violated in a typical short-range potential, as e.g. for $f(x) = e^{-x}$ when $A(x) = \frac{1}{2}e^{-2x} > 0$. We therefore conclude that the full replica symmetry breaking can not occur in short-range models, neither in the $\mu-$ dominated, nor in $R-$dominated regimes, and should therefore be replaced by a different scheme. We will discuss the necessary modifications in the next section. As to the long-ranged potentials, the scheme is fully legitimate and solutions to the above equations can be easily found for any value of $\gamma \in (0,1)$. In the $\mu-$dominated regime they are given by following expressions:

$$q_0 = \frac{\mu^{\frac{2}{\gamma-2}}}{(g^2\gamma)^{\frac{1}{\gamma-2}}(1-\gamma)^{\frac{\gamma-1}{\gamma-2}}}, \quad q_0 = q_d - \left(\frac{\mu^2}{g^2\gamma(1-\gamma)}\right)^{\frac{1}{\gamma-2}}, \quad q_k = q_d - \left(\frac{T^2}{g^2\gamma(1-\gamma)}\right)^{\frac{1}{\gamma}} \qquad (60)$$

and

$$x(q) = \frac{T}{2}\frac{2-\gamma}{[g^2\gamma(1-\gamma)]^{\frac{1}{2}}}\frac{1}{(q_d-q)^{\frac{\gamma}{2}}}, \quad q_0 \le q \le q_k\,. \qquad (61)$$

A few more useful relations follow from those above:

$$q_d = (2-\gamma)q_0, \quad x(q_0) = \frac{T}{2}\frac{2-\gamma}{\mu^{\frac{\gamma}{\gamma-2}}}\left[g^2\gamma(1-\gamma)\right]^{\frac{1}{\gamma-2}}, \quad \text{and} \quad x(q_k) = 1-\frac{\gamma}{2}\,. \qquad (62)$$

According to the general procedure such a solution makes sense as long as $q_k \ge q_0$, and it is easy to check that the condition can be rewritten as $T \le T_{AT}$, where T_{AT} is precisely the de-Almeida-Thouless temperature for this model given by Eq.(37).

A curious feature of the above solution is that the values of $q_0, x(q_k)$ and q_d turn out to be temperature-independent everywhere in the phase with broken replica symmetry. As q_d is nothing else but the thermodynamic expectation value of the collective displacement in the original model: $q_d = \frac{1}{N}\left\langle\overline{\mathbf{x}^2}\right\rangle$, it is conventional to speak in this case about "freezing" of the system below the AT line. On the other hand, the same feature ensures that in this "frozen" or glassy phase the boundary line $T_b(\mu)$ in the (μ, T) plane separating the $\mu-$dominated regime from the $R-$dominated one must be vertical. Indeed, it is given by

the condition $q_d = R^2$ which together with Eq.(60) simply amounts to $\mu = \mu_0$, with the value of μ_0 given by Eq.(35). Everywhere in the $R-$dominated glassy phase we have

$$x(q) = \frac{T}{2}\frac{2-\gamma}{[g^2\gamma(1-\gamma)]^{\frac{1}{2}}}\frac{1}{(R^2-q)^{\frac{\gamma}{2}}}, \quad q_0 \le q \le q_k\,, \tag{63}$$

with

$$q_0 = \frac{R^2}{2-\gamma}, \quad q_k = R^2 - \frac{T^{\frac{2}{\gamma}}}{[g^2\gamma(1-\gamma)]^{\frac{1}{\gamma}}}\,.$$

These observations complete our investigation of the phase diagram for the case of long-ranged correlations, full version of which was presented for the particular case $\gamma = 1/2$ in fig.1a.

Now we consider the case of logarithmic correlations, Eq.(7) assuming $R > a$. As for such a potential $A(x) \equiv 0$ this implies $\frac{d}{dq}x(q) = 0$ for $q_0 \le q \le q_k$. Indeed, our general relations Eqs.(53),(54),(56) and Eq.(57) give in this case an especially simple solution

$$x(q) = \frac{T}{g}, \quad q_0 \le q \le q_k;\ \text{where } q_0 = \frac{g}{\mu};\ q_k = 2\frac{g}{\mu} - \frac{a^2}{1-T/g} \quad \text{and} \quad q_d = 2\frac{g}{\mu} - a^2. \tag{64}$$

for the $\mu-$dominated regime. The consistency condition $q_k \ge q_0$ is satisfied as long as $T \le T_{AT}$, where T_{AT} is as expected the de-Almeida-Thouless temperature. For this case T_{AT} is given by Eq.(38). The condition $q_d = R^2$ which defines the boundary $T_b(\mu)$ with the $R-$dominated glassy regime again amounts to $\mu = \mu_0 = 2g/(R^2 + a^2)$, with the same value of μ_0 we found earlier for this case. Finally, everywhere in the $R-$dominated glassy phase occupying the rectangle $T \le T_0 = g\frac{R^2-a^2}{R^2+a^2}$, $\mu \le \mu_0$ in the (μ, T) plane we have:

$$x(q) = \frac{T}{g}, \quad q_0 \le q \le q_k;\ \text{where } q_0 = \frac{R^2+a^2}{2},\ q_k = R^2 - a^2\,\frac{T/g}{1-T/g}\,. \tag{65}$$

Using these expressions one can find the corresponding equilibrium free energy F_∞. The latter takes especially simple form for zero temperature:

$$F_\infty|_{T=0} = \begin{cases} -\frac{g^2}{2\mu a^2} \quad \text{for} \quad \mu \ge \mu_{AT} = g/a^2\,, \\ -\frac{\mu}{2}a^2 - g\ln\left(\frac{g}{\mu a^2}\right) \quad \text{for} \quad \mu_0 \le \mu \le \mu_{AT}\,, \\ \frac{\mu}{2}R^2 - g\left[1 + \ln\frac{(R/a)^2+1}{2}\right] \quad \text{for} \quad 0 \le \mu \le \mu_0 = 2g/(R^2+a^2)\,. \end{cases} \tag{66}$$

Here the upper line corresponds to the $\mu-$dominated RS regime, second line to the $\mu-$dominated solution with broken RS symmetry, and third line to $R-$dominated solution with broken RS, respectively.

The corresponding phase diagram was presented in fig.1b.

4.2 Short-ranged correlated potentials: solution within the one-step replica symmetry breaking.

In this section we will discuss the solution of our problem pertinent for the potentials with the correlation function $f(x)$ corresponding to the uniformly positive values of the function $A(x)$ defined in Eq.(59). As we argued above, this situation is typical for the short-ranged correlated potentials.

We have seen that searching for a solution within the FRSB scheme when $A(x) > 0$ leads to a contradiction. The only remaining possibility within the Parisi hierarchical ansatz is therefore to assume that the number k of steps in the replica breaking hierarchy is finite: $1 \le k < \infty$. We shall see, adopting for our model the line of reasoning suggested first by Crisanti and Sommers in [30], that the condition $A(x) > 0$ forces us to select the value $k = 1$ as the only possible. This situation is conventionally called the one-step replica symmetry breaking (1RSB). In the Appendix B we also sketch the stability analysis showing that 1RSB solution is indeed stable versus small fluctuations in the full replica space.

In the case of a general finite k the function $x(q)$ is again a non-negative and non-decreasing, but has only a finite number of points of growth in the interval $[q_0, q_k]$, parametrised by the sequences of q_i and $m_i \in [0,1]$. The variation of the free energy with respect to those parameters is again given formally by the same formula Eq.(48) but with understanding that

$$\delta x(q) = \sum_{l=0}^{k}(\delta m_{l+1} - \delta m_l)\theta(q-q_l) - \sum_{l=0}^{k}(m_{l+1}-m_l)\delta(q-q_l)\delta q_l\,. \tag{67}$$

The corresponding part of the variation of the free energy is therefore given by

$$\delta F = \frac{1}{2}\sum_{l=0}^{k}(\delta m_{l+1} - \delta m_l)\int_{q_i}^{q_k} S(q)\,dq - \frac{1}{2}\sum_{l=0}^{k}(m_{l+1}-m_l)\,S(q_i)\delta q_l\,,$$

where the function $S(q)$ is defined in Eq.(49). As $\delta m_0 = \delta m_{k+1} = 0$, the above expression can be represented in the form

$$2\delta F = \sum_{l=1}^{k}\delta m_l\int_{q_{l-1}}^{q_l} S(q)\,dq - \sum_{l=0}^{k}(m_{l+1}-m_l)\,S(q_i)\delta q_l\,, \tag{68}$$

showing that the stationarity conditions amount to the system of equations

$$\int_{q_{l-1}}^{q_l} S(q)\,dq = 0 \text{ for } l = 1,2,\ldots,k-1, \text{ and } S(q_l) = 0 \text{ for } l = 0,1,\ldots,k\,. \tag{69}$$

As the function $S(q)$ is obviously continuous in $[q_0, q_k]$, the first type of condition ensures that it takes both positive and negative values in each of the intervals $[q_{l-1}, q_l]$. According to the second condition $S(q)$ vanishes at both ends of those intervals, therefore

this function must have at least one maximum and at least one minimum in each of the above intervals. Denoting position of those extrema as q_e, we must have $\frac{d}{dq}S(q)|_{q_e} = 0$ for each point of extremum. Then differentiating Eq.(49) we get after a simple algebra the equation whose solutions should give us all possible positions of extrema q_e:

$$q_d - q_k + \int_{q_e}^{q_k} x(\tilde{q})\, d\tilde{q} = \frac{T}{\sqrt{f''(q_d - q_e)}}, \quad \forall q_e \in [q_0, q_k]\,. \tag{70}$$

Now, it is convenient to consider both the right- and left-hand side of this relation as some functions of the variable q_e. Taking the derivative over q_e from the left-hand side once gives $-x(q_e) < 0$, showing that the left-hand side is a decreasing function of its argument. Next differentiation yields $-\frac{d}{dq_e}x(q_e) \leq 0$, which shows that the left hand side is a *concave* decreasing function. Now we treat the right-hand side of Eq.(70) in the same way. First differentiation shows that the right-hand side is also a decreasing function of q_e due to the condition $f'''(x)$. At the same time its second derivative turns out to be equal to

$$\frac{T}{2}\, A(q_d - q_e)\, \frac{1}{[f''(q_d - q_e)^{5/2}} > 0$$

due to the condition $A(x) > 0$. Thus we see that the right-hand side is a *convex* decreasing function of q_e. As any convex decreasing function can have at most two points of intersection with a concave decreasing one, there is no more than two different solutions of the Eq.(70) for q_e. Hence, there could be only a single interval $[q_0, q_1]$ in the description of the function $x(q)$. Using henceforth the notation $m_1 \equiv m$, we see that such a function is given simply by

$$x(q) = \begin{cases} 0, & q < q_0 \\ m, & q \in [q_0, q_1] \\ 1, & q > q_1 \end{cases} \quad . \tag{71}$$

This is precisely the one-step replica symmetry breaking (1RSB) hierarchical ansatz, which is thus shown to be the only possibility within the Parisi scheme for systems satisfying $A(x) > 0$.

In the rest of this section we study the phase diagram resulting from the implementation of such a 1RSB scheme in our model. Instead of using the general stationarity conditions Eq.(69), we prefer to start directly with the variational free energy functional depending on the parameters q_0, q_1, q_d and $m \in [0, 1]$. In fact, we find it more convenient to use the set of variables

$$m;\; q_0;\; Q = q_1 - q_0;\; \text{and } y = q_d - q_1$$

as independent variational parameters. Substituting Eq.(71) into Eq.(47) and using the above notations we arrive at the variational free energy of the form

$$F_\infty^{1RSB} = \frac{\mu}{2}(q_0 + y + Q) - \frac{T}{2}\ln(2\pi e) - \frac{T}{2}\left(1 - \frac{1}{m}\right)\ln y - \frac{T}{2m}\ln(y + m\,Q)$$
$$-\frac{T}{2}\frac{q_0}{y + m\,Q} + \frac{1}{2T}\left[(1-m)f(y) + m\,f(y+Q) - f(0)\right]\,. \tag{72}$$

One should also remember that in the $\mu-$dominated regime the found solution should respect the inequality

$$q_0 + y + Q = q_d \le R^2 \, . \tag{73}$$

In the $R-$dominated regime the above inequality must be replaced with the equality, and the arising constraint reduces the number of independent variational parameters to three.

Let us start first with the $\mu-$dominated regime. The parameter q_0 enters in the functional linearly, and can be immediately excluded in favour of the relation

$$y = y_m(Q) = \frac{T}{\mu} - mQ \tag{74}$$

This fact allows one to write down the variational free energy as the function of only two variables, m and Q. It is also natural to operate with the so-called excess free energy given by the difference $\Delta F = F_\infty^{1RSB} - F_\infty^{RS}$ between the free energy value of the RS solution, Eq.(39) and the variational $1RSB$ free energy, Eq.(72). After simple manupulations we obtain

$$\begin{aligned}\Delta F = & \frac{\mu}{2}(1-m)Q + \frac{T}{2}\frac{1-m}{m}\ln\left(1-\frac{\mu}{T}mQ\right) \\ & + \frac{1}{2T}\left[(1-m)f\left(\frac{T}{\mu}-mQ\right) + m\, f\left(\frac{T}{\mu}+(1-m)Q\right) - f\left(\frac{T}{\mu}\right)\right] \, .\end{aligned} \tag{75}$$

Note that for $m=1$ the difference ΔF vanishes, since this choice obviously brings us back to the replica-symmetric solution. Differentiating Eq.(75) over Q and assuming $(1-m) \neq 0$ we obtain the first stationarity condition

$$Q = \frac{y_m(Q)}{\mu T}\left(f'\left[y_0(Q)\right] - f'\left[y_m(Q)\right]\right) \, , \tag{76}$$

where $y_m(Q)$ is the combination from the right-hand side of Eq.(74), and we introduced also the notation $y_0(Q) = y_m(Q) + Q$. Similarly differentiation of Eq.(75) over m yields another equation

$$-\frac{T}{m^2}\ln\left[\frac{\mu}{T}y_m(Q)\right] = Q\left(\frac{\mu}{m} + \frac{1}{T}f'\left[y_0(Q)\right]\right) - \frac{1}{T}\left(f\left[y_0(Q)\right] - f\left[y_m(Q)\right]\right) . \tag{77}$$

This set of equations determines the system behaviour in the $\mu-$dominated regime of the phase with one-step broken replica symmetry. Finally, to find the transition line $T_b(\mu)$ to $R-$dominated regime one needs to check the condition Eq.(73). For this one should be able to express the value of q_0 in terms of Q and m. To this end we notice that the stationarity condition with respect to Q at the level of the original free energy expression Eq.(72) together with Eq.(74) immediately produces the required relation:

$$q_0 = -\frac{1}{\mu^2}f'\left[y_0(Q)\right] \, . \tag{78}$$

Before discussing features of the resulting phase diagram in full generality it is worth pointing out the existence of a particular case when the system of equations Eqs.(76,77) and Eq.(74) allows for an explicit algebraic solution. This is precisely the case of logarithmically correlated potential, Eq.(7). Indeed, a direct verification shows that the substitution

$$Q = \frac{g}{\mu} - \frac{a^2}{1 - T/g}, \quad m = \frac{T}{g}\,. \tag{79}$$

solves all those equations. This observation should look less surprising if one comes back to the solution Eq.(64) found earlier as a particular limiting case of the full replica symmetry breaking ansatz. One then realizes that the corresponding function $x(q)$ was constant in the interval $q \in [q_0, q_k]$, the feature being characteristic rather of one-step replica symmetry breaking. This is just another evidence towards marginal nature of the potential with logarithmic correlations: the case can be looked at both as a limiting special case of FRSB solution, and that of 1RSB solution.

After this digression, we proceed to discussing typical features of the phase diagram in a generic case of one-step replica symmetry breaking. As we have already seen, the transition to the phase with broken replica symmetry may occur in two different ways: either along the line $Q \to 0$, or along the line $m \to 1$. Let us first investigate the behaviour of the free energy difference (75) for small $Q \ll 1$. Expanding that expression to up to the first two nonvanishing terms gives

$$\Delta F = \frac{m(1-m)}{2T} \left\{ \frac{Q^2}{2} \left[-\mu^2 + f''(T/\mu) \right] + \frac{Q^3}{3} \left[-m\frac{\mu^3}{T} + (\frac{1}{2} - m) f'''(T/\mu) \right] \right\}. \tag{80}$$

Maximization with respect to Q shows that $Q = 0$ for $\mu^2 \geq f''(T/\mu)$ which is precisely the de-Almeida-Thouless condition Eq.(31) of stability of the replica-symmetric solution. Below the AT line the maximum of the excess free energy happens at a nonzero value of Q given by:

$$Q = \frac{f''(T/\mu) - \mu^2}{-m\frac{\mu^3}{T} + (\frac{1}{2} - m) f'''(T/\mu)}. \tag{81}$$

This allows one to write ΔF in terms of m only as

$$\Delta F = \frac{m(1-m)}{6T} \frac{\left[-\mu^2 + f''(T/\mu)\right]^3}{\left[m\frac{\mu^3}{T} - (\frac{1}{2} - m) f'''(T/\mu)\right]^2}\,. \tag{82}$$

We see that the excess free energy grows cubically in the glassy phase in the vicinity of the AT line. Such a behaviour is typical for continuous glass transitions to a phase with broken replica symmetry.

Requiring maximum of this excess free energy with respect to m one finds the equilibrium value of this parameter in the vicinity of the AT line:

$$m = -\frac{1}{2}\frac{T}{\mu^3} f'''\left(\frac{T}{\mu}\right)\,. \tag{83}$$

Such value of m may tend to the maximal possible value $m = 1$ when the temperature T and the parameter μ approach along the AT line a point (μ_m, T_m) where they satisfy the system of two equations:

$$\frac{\mu_m^3}{T_m} + \frac{1}{2} f'''\left(\frac{T_m}{\mu_m}\right) = 0, \quad -\mu_m^2 + f''\left(\frac{T_m}{\mu_m}\right) = 0\,. \tag{84}$$

It is easy to check that these conditions are precisely those ensuring that (μ_m, T_m) is the point of a maximum of the AT line, i.e. $\frac{dT_{AT}}{d\mu} = 0$, $\frac{d^2 T_{AT}}{d\mu^2} < 0$. Close to the point of maximum the AT line is described by

$$\tau_{AT} = \frac{3\delta^2}{2}\left[1 - \frac{T_m^2}{6\mu_m^4} f''''\left(\frac{T_m}{\mu_m}\right)\right], \quad \delta \geq 0, \tag{85}$$

where $\tau_{AT} = (T_m - T_{AT})/T_m \ll 1,\ \delta = (\mu - \mu_m)/\mu_m \ll 1$.

For the point of maximum to happen within the $\mu-$dominated regime the corresponding value μ_m should obviously satisfy $\mu_m \leq \mu_0$, with μ_0 being the value earlier defined by Eqs.(29,30). This condition after a simple algebra reduces to the inequality

$$R^2 \geq R_*^2 = -2\frac{f''(\tau_*)}{f'''(\tau_*)} - \frac{f'(\tau_*)}{f''(\tau_*)} \tag{86}$$

for the confinement radius R. Here τ^* is the (unique) solution of the equation $\tau_* = -2\frac{f''(\tau_*)}{f'''(\tau_*)} \equiv z(\tau_*)$. Existence (and uniqueness) of such a solution is ensured for the short-ranged potentials satisfying our main condition $A(x) > 0$. Indeed, in that case the function $z(\tau)$ satisfies $z(0) > 0$ and $dz/d\tau = 1 - 2A(\tau)/[f'''(\tau)]^2 < 1$, so that the point of intersection of the graph of the function $y = z(\tau)$ and the straight line $y = \tau$ is obviously unique.

As long as $R_{cr} \leq R \leq R_*$ the position of the maximum on the AT line is irrelevant as the AT line earlier meets the line $T_b(\mu)$ at the point (μ_0, T_0). As we have already seen in our analysis of the replica-symmetric solution, the line $T_b(\mu)$ must have at the meeting point the vertical tangent:$\frac{d}{d\mu}T_B(\mu)|_{\mu_0} = \infty$. The line $T_b(\mu)$ then continues to the phase with broken replica symmetry. The corresponding equation is given by solving the system of equations Eq.(76) and (77) together with the condition:

$$y_0(Q) - \frac{1}{\mu^2} f'\left[y_0(Q)\right] = R^2\,. \tag{87}$$

The resulting transition line $T_b(\mu)$ evidently ends up at zero temperature $T = 0$ at some point μ_b of the $\mu-$axis. A typical phase diagram of this sort was presented in fig.1c for the particular case of $f(x) = e^{-x}$, when the value R_* can be found analytically as $R_* = \sqrt{3}$, see Eq.(86). Note that in contrast to the long-ranged case now the line $T_b(\mu)$ is not strictly vertical inside the glassy phase, although actual numerical values of its derivative are quite big. Let us also note that in the $R-$dominated part of the glassy phase (i.e. to

the left of the line $T_b(\mu)$) the values of the parameters m, Q, q_0 at a given temperature T "freeze" to their $\mu-$independent values which those parameters acquired precisely at the point of the transition line $T_b(\mu)$ such that $T_b = T$.

It is appropriate to recall that the AT stability line Eq.(31) for a short-range potential meets $T = 0$ line at the point $\mu = \mu_{AT} = [f''(0)]^{1/2}$. Therefore for such systems a non-trivial transition to the phase with broken replica symmetry is possible also at zero temperature with decreasing μ, in contrast to the case of long-ranged potentials. In fact, it is known that some aspects of zero-temperature behaviour may be amenable to investigation without resorting to replicas, see [9, 11], so it may be useful to provide below a more detailed picture of the zero-temperature transition within the replica approach. In performing the zero-temperature limit one first should realize that $\lim_{T\to 0} m/T = v < \infty$, so that the relevant parameters governing the system behaviour in that case will be v and Q. The variational excess (free) energy is given in terms of these parameters by

$$2\Delta F_{T=0} = \mu\, Q + \frac{1}{v}\ln\left(1 - \mu\, v\, Q\right) + v\,\left[f(Q) - f(0) - Qf'(0)\right]. \tag{88}$$

The corresponding stationarity conditions can be reduced after some algebra to the set of two equations:

$$\frac{\mu^2\, Q}{1 - \mu\, v\, Q} = f'(Q) - f'(0), \tag{89}$$

$$\frac{1}{v}\ln\left(1 - \mu\, v\, Q\right) = -\mu\, Q + v\left[f(Q) - f(0) - Qf'(Q)\right], \tag{90}$$

which also could be obtained by performing the limit $T \to 0$ directly in Eq.(76) and (77). A straightforward calculation shows that close to the transition point $\mu = \mu_{AT} = [f''(0)]^{1/2}$ the parameter v tends to the value $v = -\frac{f'''(0)}{2\mu_{AT}^3}$, and the corresponding excess energy at zero temperature behaves again cubically in the vicinity of transition (cf. Eq.(82):

$$\Delta\, F_{T=0} = -\frac{2}{3}\frac{\mu_{AT}^3}{f'''(0)}\left(\frac{\mu}{\mu_{AT}} - 1\right)^3 > 0\,. \tag{91}$$

Let us now consider the case of infinite confinement radius $R \to \infty$, which was the subject of earlier studies within GVA [13]. In this situation it is interesting to investigate the behaviour of the system at zero temperature and vanishing confining potential $\mu \to 0$. Analysis of the system Eq.(89) then shows that as long as $\mu \to 0$ necessarily $Q \to \infty$ in such a way that $\mu v Q \to 1$, provided for large Q holds $f(Q) \to 0$ as well as $Qf'(Q) \to 0$. One then finds after some algebra that the parameter v must satisfy in this limit the equation

$$v\, e^{-f(0)v^2} = -\frac{\mu}{f'(0)}\,.$$

The valid solution $v = v_\mu \gg 1$ is given asymptotically by

$$v_\mu^2 = -\frac{1}{f(0)}\ln\left(-\frac{\mu}{f'(0)}\right) + O(\ln|\ln\mu|)\,. \tag{92}$$

This in turn yields for the parameter Q and for the excess free energy the following asymptotic expressions:

$$Q \approx \frac{1}{\mu}\frac{1}{\sqrt{-\frac{1}{f(0)}\ln\left(-\frac{\mu}{f'(0)}\right)}}, \quad \Delta F \approx -\sqrt{f(0)|\ln\left(-\frac{\mu}{f'(0)}\right)|}\,. \tag{93}$$

Recalling that for zero temperature $q_d = Q - \frac{1}{\mu^2}f'(Q)$, hence $Q \to \infty$ implies $q_d \approx Q$, we indeed see that Eq.(93) agrees with one obtained in the framework of GVA[13] for the mean displacement parameter $q_d = \frac{1}{N}\langle \mathbf{x}^2\rangle$. One also can perform similar calculations in the opposite limiting case $\mu = 0$ and $R \to \infty$, the result for $T = 0$ being given by the same asymptotic formulae Eq.(93) with the only replacement $\mu \to 1/R^2$.

Returning to considerations of general finite values of the parameter R, we see that the $\mu-$dominated regime described at $T = 0$ by the equations Eq.(89) can exist as long as

$$Q - \frac{1}{\mu^2}f'(Q) \leq R^2\,, \tag{94}$$

where we took into account the zero-temperature limit of Eq.(78). With decreasing μ the latter condition is first violated at some value $\mu = \mu_b$ which can be found numerically by solving the system of three equations: Eq.(89,90) and (94). For lower values of μ, i.e. for $\mu < \mu_b$, the parameters Q and v will "freeze" at their boundary values $v = v(\mu_b)$ and $Q = Q(\mu_b)$.

The shape of the phase diagram described above as typical for $R < R_*$, see Eq.(86), experiences essential modifications as long as R exceeds R_*. In the latter case the AT Temperature line $T_{AT}(\mu)$ starts to decrease with decreasing μ to the left of the maximum: $\mu < \mu_m$. We shall see that the correct form of the phase diagram in this situation requires considering the transition line to the phase with broken replica symmetry given by the condition $m \to 1$.

To investigate such a possibility we expand the excess free energy Eq.(75) in powers of $p = 1 - m$ as

$$\Delta F = A(Q)\,p + B(Q)\,p^2 + C(Q)\,p^3 + \ldots, \tag{95}$$

Extremising with respect to p gives the condition $A(Q) + 2p\,B(Q) + 3p^2\,C(Q) + \ldots = 0$ so that the equilibrium value of p for small p is approximately given by $p_m \approx -A(Q)/2B(Q) + \ldots$. The excess free energy in the phase with broken replica symmetry is then given by $\Delta F \approx A(Q)\,p_m + B(Q)p_m^2 = -A(Q)^2/4B(Q) \geq 0$, which implies $B(q) \leq 0$. As $p_m \geq 0$, we conclude $A(Q) \geq 0$. It is evident also that the line at which $p_m \to 0$ implies the condition $A(Q) = 0$. Moreover, extremum conditions of Eq.(95) with respect to Q implies in the limit $p_m \to 0$ another condition $A'(Q) = 0$. This consideration shows that the line at which $m \to 1$ within the phase with broken replica symmetry is given by the system of equations:

$$A(Q) = 0, \quad \frac{d}{dQ}A(Q) = 0, \tag{96}$$

where explicit expression for $A(Q)$ can be found from the condition $A(Q) = \frac{\partial \Delta F}{\partial p}|_{p=0}$. This gives, up to an unessential factor

$$A(Q) = \frac{\mu}{T} Q + \ln\left(1 - \frac{\mu}{T} Q\right) + \frac{1}{T^2}\left[f\left(\frac{T}{\mu} - Q\right) - f\left(\frac{T}{\mu}\right) + Q f'\left(\frac{T}{\mu}\right)\right] \tag{97}$$

and therefore

$$A'(Q) = -\frac{\mu^2}{T^2}\frac{Q}{1 - \frac{\mu}{T}Q} - \frac{1}{T^2}\left[f'\left(\frac{T}{\mu} - Q\right) - f'\left(\frac{T}{\mu}\right)\right]. \tag{98}$$

Substituting these expressions to equations Eq.(96) we see that the resulting conditions are indeed equivalent to the pair of equations (76) and (77), provided $m \to 1$. We will be looking for its non-vanishing solution $Q = Q_* > 0$.

The equation Eq.(98) for $Q \to 0$ reads $A'(Q) \approx \frac{Q}{T^2}\left[-\mu^2 + f''(T/\mu)\right]$, hence $A'(Q)$ is positive just below the AT instability line Eq.(31). We may expect this sign to change in the vicinity of the point of maximum (μ_m, T_m) from Eq.(84), as we know that the line $m = 1$ passes through that point. In the vicinity of AT line the parameter Q is small, and we can expand

$$A(Q) = -\frac{Q^2}{2T^2}\left[\mu^2 - f''\left(\frac{T_m}{\mu_m}\right)\right] - \frac{1}{3}\frac{Q^3}{T^2}\left[\frac{\mu^3}{T} + \frac{1}{2}f'''\left(\frac{T}{\mu}\right)\right] + \frac{Q^4}{4T^2}\left[\frac{1}{6}f''''(T/\mu) - \frac{\mu^4}{T^2}\right] + \ldots \tag{99}$$

One indeed finds from this expression a non-vanishing solution $Q_* > 0$ of the system Eq.(96) for $T < T_m, \mu < \mu_m$ close to the point of maximum to be

$$Q_* = \frac{2}{3}\frac{\frac{\mu^3}{T} + \frac{1}{2}f'''\left(\frac{T}{\mu}\right)}{\frac{1}{6}f''''\left(\frac{T_m}{\mu_m}\right) - \frac{\mu^4}{T^2}}, \quad T - T_m \ll T_m,\ \mu - \mu_m \ll \mu_m \tag{100}$$

Substituting this back to Eq.(97) we get an explicit relation between T and μ defining the line $T_1(\mu)$ along which a solution with broken replica symmetry $Q \neq 0$ appears first with $m = 1$ close to (μ_m, T_m):

$$\left[\mu^2 - f''\left(\frac{T_m}{\mu_m}\right)\right]\left[\frac{1}{6}f''''\left(\frac{T_m}{\mu_m}\right) - \frac{\mu^4}{T_1^2}\right] + \frac{2}{9}\left[\frac{\mu^3}{T_1} + \frac{1}{2}f'''\left(\frac{T_1}{\mu}\right)\right]^2 = 0, \tag{101}$$

Expanding in powers of $\tau_1 = (T_m - T_1)/T_m \ll 1,\ \delta = (\mu_m - \mu)/\mu_m \ll 1$ one can find to the leading order the equation for this line to be

$$\tau_1 = \frac{\delta^2}{2}\left[1 - \frac{T_m^2}{6\mu_m^4}f''''\left(\frac{T_m}{\mu_m}\right)\right], \quad \delta \leq 0, \tag{102}$$

which shows that the transition line $T_1(\mu)$ when approaching the point (μ_m, T_m) from the left has in that point its maximum. At the point of maximum it therefore smoothly meets

the AT line which in the same approximation is described by Eq.(85), with only second derivative experiencing a jump.

For lower values of μ away from μ_c the temperature $T_1(\mu)$ decreases down to a point where that line meets the boundary line $T_b(\mu)$, and the system transits to $R-$dominated regime. The point of intersection can be found only numerically. For even lower values of μ the transition temperature $T_1(\mu)$ freezes to its value at the intersection point, see fig.4. In the limiting case of infinite confinement radius $R = \infty$ the $\mu-$dominated regime covers the whole phase diagram, and it makes sense to investigate the asymptotic behaviour of the line $T_1(\mu)$ for $\mu \to 0$. Anticipating the behaviour $T_1|_{\mu\to 0} \to 0$ in such a way that $T_1 >> \mu$ one can show that the corresponding non-vanishing solution $Q_* > 0$ of the system Eq.(96) must be sought close to its maximal possible value $Q_* = Q_*^{max} = T/\mu$. Using $f(\infty) = f'(\infty) = 0$ one finds after some algebra the corresponding asymptotic expression for the transition line $T = T_1(\mu)$:

$$\mu = -\frac{f'(0)}{T} \exp -\frac{f(0)}{T^2}, \quad T \to 0\,. \tag{103}$$

For completeness, we present in fig.4 also typical phase diagrams for systems with short-range correlations in two limiting cases: in (T, μ) plane for $R = \infty$ and in (T, R) plane for $\mu = 0$. Recall, that in the latter case the de Almeida-Thouless temperature for our choice of the correlation function is explicitly given by $T_{AT} = T_0(R) = (R^2 - 1)e^{-\frac{1}{2}(R^2-1)}$.

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A Appendix: Parisi matrix, its eigenvalues and evaluation of traces in the replica limit.

We start with describing the well-known structure of the $n \times n$ matrix Q in the Parisi parametrisation. At the beginning we set n diagonal entries $q_{\alpha\alpha}$ all to the same value $q_{\alpha\alpha} = 0$. This value will be maintained at every but last step of the recursion. The off-diagonal part of the matrix Q in the Parisi scheme is built recursively as follows. At the first step we single out from the $n \times n$ matrix Q the chain consisting of n/m_1 blocks of the size $m_1 \leq n$, each situated on the main diagonal. All off-diagonal entries $q_{\alpha\beta}$, $\alpha \neq \beta$ inside those blocks are filled in with the same value $q_{\alpha\beta} = q_1 \leq 0$, whereas all the remaining $n^2(1 - 1/m_1)$ entries of the matrix Q are set to the value $0 < q_0 \leq q_1$.

The latter entries remain from now on intact to the end of the procedure, whereas some entries inside the diagonal $m_1 \times m_1$ blocks will be subject to a further modification. At the next step of iteration in each of those diagonal blocks of the size m_1 we single out the chain of m_2/m_1 smaller blocks of the size $m_2 \leq m_1$, each situated on the main diagonal. All off-diagonal entries $q_{\alpha\beta}$, $\alpha \neq \beta$ inside those sub-blocks are filled in with the same value $q_{\alpha\beta} = q_2 \geq q_1$, whereas all the remaining entries of the matrix Q hold their old values. At the next step only some entries inside diagonal blocks of the size m_2 will be modified., etc. Iterating this procedure step by step one obtains after k steps a hierarchically built structure characterized by the sequence of integers

$$n = m_0 \geq m_1 \geq m_2 \geq \ldots \geq m_k \geq m_{k+1} = 1 \qquad (104)$$

and the values placed in the diagonal blocks of the Q matrix satisfying:

$$0 < q_0 \leq q_1 \leq q_2 \leq \ldots \leq q_k \qquad (105)$$

Finally, we complete the procedure by filling in the n diagonal entries $q_{\alpha\alpha}$ of the matrix Q with one and the same value $q_{\alpha\alpha} = q_d \geq q_k$.

For the subsequent analysis we need the eigenvalues of the Parisi matrix Q. Those can be found easily together with the corresponding eigenvectors built according to a recursive procedure which uses the sequence Eq.(104). It is convenient to visualize eigenvectors as being "strings" of n boxes numbered from 1 to n, with l^{th} component being a content of the box number l.

At the first step $i = 1$ we choose the eigenvector to have all n boxes filled with the same content equal to unity. The corresponding eigenvalue is non-degenerate and equal to

$$\lambda_1 = q_d + q_k(m_k - 1) + q_{k-1}(m_{k-1} - m_k) + \ldots + q_1(m_1 - m_2) + q_0(m_0 - m_1) \qquad (106)$$

Now, at the subsequent steps $i = 2, 3, \ldots, k+2$ one builds eigenvectors by the following procedure. The string of n boxes of an eigenvector belonging to i^{th} family are subdivided into n/m_{i-1} substrings of the length m_{i-1}, and numbered accordingly by the index $j = 1, 2, \ldots, n/m_{i-1}$. All m_{i-1} boxes of the first substring $j = 1$ are filled invariably with all components equal to 1. Next we fill m_{i-1} boxes in one (and only one) of the remaining $\frac{n}{m_{i-1}} - 1$ substrings with all components equal to -1. In doing so we however impose a constraint that the substrings with the indices j given by $j = 1 + l\frac{m_{i-2}}{m_{i-1}}$ should be excluded from the procedure, with l being any integer satisfying $1 \leq l \leq \frac{n}{m_{i-2}} - 1$. After the choice of a particular substring is made, we fill all $n - 2m_{i-1}$ boxes of the remaining substrings with identically zero components. It is easy to see that all $d_i = n/m_{i-1} - n/m_{i-2}$ different eigenvectors of i^{th} family built in such a way correspond to one and the same d_i-degenerate eigenvalue

$$\lambda_i = q_d + q_k(m_k - 1) + q_{k-1}(m_{k-1} - m_k) + \ldots + q_{i-1}(m_{i-1} - m_i) - q_{i-2}(m_{i-1}) \qquad (107)$$

In this way we find all n possible eigenvalues, the last being equal to

$$\lambda_{k+2} = q_d - q_k\, m_{k+1} \equiv q_d - q_k. \tag{108}$$

The completeness of the procedure follows from the fact that sum of all the degeneracies d_i is equal to

$$1 + \left(\frac{n}{m_1} - 1\right) + \left(\frac{n}{m_2} - \frac{n}{m_1}\right) + \ldots + \left(\frac{n}{m_{k+1}} - \frac{n}{m_k}\right) = n$$

Note that all the found eigenvalues are positive due to inequalities Eq.(105) between various q_i, which is required by the positive definiteness of the matrix Q. Note also that all eigenvectors built in this way are obviously linearly independent, although the eigenvectors belonging to the same family are not orthogonal. The latter fact however does not have any bearing for our considerations.

To facilitate the subsequent treatment it is convenient to introduce the following (generalized) function of the variable q:

$$x(q) = n + \sum_{l=0}^{k} (m_{l+1} - m_l)\, \theta(q - q_l) \tag{109}$$

where we use the notation $\theta(z)$ for the Heaviside step function: $\theta(z) = 1$ for $z > 0$ and zero otherwise. In view of the inequalities Eq.(104,105) the function $x(q)$ is piecewise-constant non-increasing, and changes between n and 1 as follows:

$$x(q < q_0) = m_0 \equiv n,\; x(q_0 < q < q_1) = m_1,\; \ldots,\; x(q_{k-1} < q < q_k) = m_k,\; x(q > q_k) = m_{k+1} \equiv 1 \tag{110}$$

Comparison of this form with Eq.(109) makes evident the validity of a useful inversion formula:

$$\frac{1}{x(q)} = \frac{1}{n} + \sum_{l=0}^{k} \left(\frac{1}{m_{l+1}} - \frac{1}{m_l}\right)\, \theta(q - q_l) \tag{111}$$

which will be exploited by us shortly.

As observed by Crisanti and Sommers[30] one can represent the eigenvalues Eq.(107) of the Parisi matrix in a compact form via the following remarkable identities:

$$\lambda_1 = \int_0^{q_d} x(q)\, dq = nq_0 + \int_{q_0}^{q_d} x(q)\, dq, \quad \lambda_{i+2} = \int_{q_i}^{q_d} x(q)\, dq, \quad i = 0, 1, \ldots, k \tag{112}$$

As a consequence, these relations imply for any analytic function $g(x)$ the identity

$$\frac{1}{n} Tr\,[g(Q)] = \frac{1}{n} \sum_{i=1}^{k+2} g(\lambda_i)\, d_i = \frac{1}{n} g\left(nq_0 + \int_{q_0}^{q_d} x(q)\, dq\right) + \sum_{l=0}^{k} \left(\frac{1}{m_{l+1}} - \frac{1}{m_l}\right) g\left(\int_{q_l}^{q_d} x(q)\, dq\right). \tag{113}$$

Next one observes that taking the derivative of the generalized function from Eq.(111) produces

$$\frac{d}{dq}\left[\frac{1}{x(q)}\right] = \sum_{l=0}^{k}\left(\frac{1}{m_{l+1}} - \frac{1}{m_l}\right)\,\delta(q-q_l). \tag{114}$$

This fact allows one to rewrite the sum in Eq.(113) in terms of an integral, yielding

$$\frac{1}{n}Tr\,[g(Q)] = \frac{1}{n}g\left(nq_0 + \int_{q_0}^{q_d} x(q)\,dq\right) + \int_{q_0-0}^{q_k+0} g\left(\int_{q}^{q_d} x(\tilde{q})\,d\tilde{q}\right)\,\frac{d}{dq}\left[\frac{1}{x(q)}\right]\,dq,$$

where the short-hand notation $q \pm 0$ designates the limit from below/above. Further performing integration by parts, and using $x(q > q_k) = 1$, $x(q < q_0) = n$, we finally arrive at

$$\begin{aligned}\frac{1}{n}Tr\,[g(Q)] &= \frac{1}{n}\left[g\left(nq_0 + \int_{q_0}^{q_d} x(q)\,dq\right) - g\left(\int_{q_0}^{q_d} x(q)\,dq\right)\right] \\ &+ \; g(q_d - q_k) + \int_{q_0}^{q_k} g'\left(\int_{q}^{q_d} x(\tilde{q})\,d\tilde{q}\right)\,dq.\end{aligned} \tag{115}$$

We are actually interested in the replica limit $n \to 0$. According to the Parisi prescription in such a limit the inequality Eq.(104) should be reversed:

$$n = 0 \le m_1 \le m_2 \le \ldots \le m_k \le m_{k+1} = 1 \tag{116}$$

and the function $x(q)$ is now transformed to a non-decreasing function of the variable q in the interval $q_0 \le q \le q_k$, and satisfying outside that interval the following properties

$$x(q < q_0) = 0, \quad \text{and} \quad x(q > q_k) = 1. \tag{117}$$

In general,such a function also depends on the increasing sequence of k parameters m_l described in Eq.(43) .

The form of Eq.(115) makes it easy to perform the limit $n \to 0$ explicitly, and to obtain after exploitation of Eq.(42) an important identity Eq.(44) helping to evaluate the traces in the replica limit. Finally, let us mention the existence of an efficient method of the "replica Fourier transform" allowing one to diagonalise (and otherwise work) with much more general types of hierarchical matrices, see [37, 38] for more detail.

B Appendix: Stability analysis of the replica solutions.

B.1 General relations.

In this part we are going to derive the general stability equations in the $\mu-$ dominated regime, and then indicate modifications arising in the $R-$ dominated case.

In general, the stability analysis amounts to expanding the function $\Phi(Q)$ in Eq.(14) around the extremum point up to the second order in deviations: $\Phi = \Phi_{SP} + \delta\Phi + \frac{1}{2}\delta^2\Phi$. In the $\mu-$ dominated regime the stationary point is inside the integration region and the stationarity condition amounts to $\delta\Phi = 0$ yielding the system (16)-(17). The term $\delta^2\Phi$ is a quadratic form in independent fluctuation variables $\delta q_{ab}, a \le b$ and can be generally written as $\delta^2\Phi = \sum_{(ab),(cd)} \delta q_{(ab)} G_{(ab),(cd)}\, \delta q_{(cd)}$. As usual the stable extremum corresponds to the positive definite quadratic form, and along the critical line the quadratic form becomes semi-definite. Checking positive definiteness of $\delta^2\Phi$ amounts to finding the (generalized) eigenvalues Λ of the matrix $G_{(ab),(cd)} = \frac{\partial^2\Phi}{\partial q_{ab}\partial q_{cd}}$, i.e. solving the equations

$$\sum_{(cd),c\le d} G_{(ab),(cd)}\, \eta_{(cd)} = \Lambda \sum_{(cd),c\le d} C_{(ab),(cd)}\, \eta_{(cd)}, \quad a \le b, \tag{118}$$

Here the notation η_{ab}, $a \le b$ is used for $n(n+1)/2$ components of a (generalized) eigenvector η and $C > 0$ can be any real symmetric positive definite matrix used to define a suitable scalar product $\left(\delta q^{(A)}, \delta q^{(B)}\right)_C$ in the space of fluctuation vectors δq, that is

$$\left(\delta q^{(A)}, \delta q^{(B)}\right)_C = \sum_{(ab),(cd)} \delta q^{(A)}_{(ab)} C_{(ab),(cd)} \delta q^{(B)}_{(cd)}. \tag{119}$$

Indeed, for any choice of C all generalized eigenvalues are real due to the symmetry properties of the matrix G , and the eigenvectors $\eta^{(i)}$ and $\eta^{(j)}$ corresponding to different eigenvalues $\Lambda_i \ne \Lambda_j$ are orthogonal with respect to the chosen scalar product: $\left(\eta^{(i)}, \eta^{(j)}\right)_C = 0$, hence linearly independent and form a basis. Expanding the fluctuations in this basis as $\delta q = \sum_i p_i \eta^{(i)}$ one then finds generically $\delta^2\Phi = \sum_j \Lambda_i p_i^2 \left(\eta^{(i)}, \eta^{(i)}\right)_C$, and the positive definiteness amounts to the condition $\Lambda_i > 0$ for all i. The instability occurs when some of the eigenvalues vanish: $\Lambda_i = 0$, and it is easy to see that the condition for the instability is independent of the choice of the matrix C in the definition of the scalar product Eq.(119). In fact, in our analysis we find it convenient to introduce a formal convention $\eta_{(ab)} = \eta_{(ba)}$ for eigenvector components, which makes it natural to think of the eigenvectors η as being real symmetric matrices. Accordingly, we find it convenient to define the scalar product in that vector space for any two such eigenvectors $\eta^{(A)}$ and $\eta^{(B)}$ as

$$\left(\eta^{(A)}, \eta^{(B)}\right) = \mathrm{Tr}\left[\eta^{(A)}\eta^{(B)}\right] \tag{120}$$

This simply corresponds to choosing the matrix C in Eq.(119) to be diagonal, with the diagonal entries given by $C_{(ab),(ab)} = (2 - \delta_{ab})$.

Our first task is to determine explicitly the entries of the stability matrix G for our problem. The structure of the replica free energy functional Eq.(14) suggests to represent the matrix G as a sum of two terms $G = G_I + G_{II}$ such that:

$$(G_I)_{(ab),(cd)} = -\frac{T}{2}\frac{\partial^2}{\partial q_{ab}\partial q_{cd}} \ln\det Q, \quad (G_{II})_{(ab),(cd)} = -\frac{1}{2T}\frac{\partial^2}{\partial q_{ab}\partial q_{cd}} \sum_{(a\ne b)} f(D_{ab}), \tag{121}$$

where the last summation is assumed to go over *all* possible pairs (ab) with $a \neq b$, and we introduced a short-hand notation $D_{ab} = \frac{1}{2}(q_{aa} + q_{bb} - 2q_{ab})$. A straightforward application of the Wick theorem shows that:

$$(G_I)_{(a\neq b),(c\neq d)} = T\left[\left(Q^{-1}\right)_{ac}\left(Q^{-1}\right)_{bd} + \left(Q^{-1}\right)_{ad}\left(Q^{-1}\right)_{bc}\right],\ (G_I)_{(aa),(cc)} = \frac{T}{2}\left[\left(Q^{-1}\right)_{ad}\right]^2, \tag{122}$$

$$(G_I)_{(a\neq b),(cc)} = T\left(Q^{-1}\right)_{ac}\left(Q^{-1}\right)_{bc},\ (G_I)_{(aa),(bc)} = T\left(Q^{-1}\right)_{ab}\left(Q^{-1}\right)_{ac}. \tag{123}$$

In view of the convention $\eta_{(ab)} = \eta_{(ba)}$ for eigenvector components, we have:

$$\sum_{(cd),c\leq d} (G_I)_{(ab),(cd)}\ \eta_{(cd)} = \frac{T}{1+\delta_{ab}} \sum_{(cd)} \left(Q^{-1}\right)_{ac}\left(Q^{-1}\right)_{bd} \eta_{(cd)}, \tag{124}$$

where δ_{ab} stands for the Kronecker symbol, and the summation in the right-hand side goes over all pairs of indices without restrictions.

Further simple differentiation gives for the entries of G_{II} with $a \neq b$ the expressions:

$$(G_{II})_{(a\neq b),(a\neq b)} = -\frac{1}{T} f''(D_{ab}),\ (G_{II})_{(a\neq b),(aa)} = (G_{II})_{(a\neq b),(bb)} = \frac{1}{2T} f''(D_{ab}), \tag{125}$$

and $(G_{II})_{(a\neq b),(cd)} = 0$ for all other choices of (cd). If however $a = b$, we similarly find

$$(G_{II})_{(aa),(aa)} = -\frac{1}{4T}\sum_{c,\, c\neq d} f''(D_{ac}),\ (G_{II})_{(aa),(ab)} = \frac{1}{2T} f''(D_{ab}),\ (G_{II})_{(aa),(bb)} = -\frac{1}{4T} f''(D_{ab}), \tag{126}$$

and $(G_{II})_{(aa),(cd)} = 0$ for all other choices of (cd). We therefore see that

$$\sum_{(cd),c\leq d} (G_{II})_{(ab),(cd)}\ \eta_{(cd)} = \frac{1}{2T} f''(D_{ab})\left[\eta_{(aa)} + \eta_{(bb)} - 2\eta_{(ab)}\right], \quad a \neq b, \tag{127}$$

and for $a = b$

$$\sum_{(cd),c\leq d} (G_{II})_{(aa),(cd)}\ \eta_{(cd)} = -\frac{1}{4T}\sum_{c,\, c\neq a} f''(D_{ac})\left[\eta_{(aa)} + \eta_{(cc)} - 2\eta_{(ac)}\right]. \tag{128}$$

Combining all these expressions and definitions we see that the system of equations Eq.(118) for eigenvector component $\eta_{(ab)}$ and eigenvalues Λ of the stability matrix G in the $\mu-$dominated regime takes the form:

$$T\sum_{(cd)} \left(Q^{-1}\right)_{ac}\left(Q^{-1}\right)_{bd} \eta_{(cd)} + \frac{1}{2T} f''(D_{ab})\left[\eta_{(aa)} + \eta_{(bb)} - 2\eta_{(ab)}\right] = 2\Lambda\, \eta_{(ab)}, \quad a \neq b \tag{129}$$

and for the eigenvector components with $a = b$:

$$\frac{T}{2}\sum_{(cd)}\left(Q^{-1}\right)_{ac}\left(Q^{-1}\right)_{ad}\eta_{(cd)} - \frac{1}{4T}\sum_{c} f''(D_{ac})\left[\eta_{(aa)} + \eta_{(cc)} - 2\eta_{(ac)}\right] = \Lambda\,\eta_{(aa)}. \tag{130}$$

Assuming $T > 0$ and introducing the notation $\Lambda^* = 2\Lambda/T$, we can rewrite the above pair in the unified form:

$$\sum_{cd}\left(Q^{-1}\right)_{ac}\eta_{(cd)}\left(Q^{-1}\right)_{db} + \frac{1}{T^2}f''(D_{ab})(\delta D_{ab}) - \frac{1}{T^2}\delta_{ab}\sum_{c} f''(D_{ac})(\delta D_{ac}) = \Lambda^*\eta_{(ab)} \tag{131}$$

where we used the notation

$$\delta D_{ab} = \frac{1}{2}\left[\eta_{(aa)} + \eta_{(bb)} - 2\eta_{(ab)}\right]. \tag{132}$$

Let us now indicate minor changes required in the similar procedure for $R-$dominated regime. First of all, in that regime the diagonal entries q_{aa} are fixed to the boundary value $q_{aa} = R^2, \forall a$. The stability for the linear deviations $\delta\Phi > 0$ implies $\partial\Phi/\partial q_{aa} < 0$ at fixed q_{ab} on the boundary, which amounts to the inequality

$$\mu/T < \sum_{b}(Q^{-1})_{ab}\,. \tag{133}$$

On the other hand, in the $\mu-$dominated regime the last inequality is replaced by equality. Hence, it is easy to see that the inequality (133) means nothing but $\mu < \mu_0$ with $T = T_b(\mu_0)$.

As to the quadratic form $\delta^2\Phi$, the entries of the matrix G involve in this regime only pairs of indices with $(a \neq b)$ and $(c \neq d)$. The explicit expressions for those entries are given by formally the same equations as in the $\mu-$dominated regime. The eigenvectors η must now have only components $\eta_{(a<b)}$. It is again convenient to introduce the convention $\eta_{(ab)} = \eta_{(ba)}, a \neq b$ for those components, together with $\eta_{(aa)} = 0, \forall\, a$. The corresponding eigen-equation then amounts to

$$\sum_{(cd)}\left(Q^{-1}\right)_{ac}\left(Q^{-1}\right)_{bd}\eta_{(cd)} - \frac{1}{T^2}f''\left(R^2 - q_{ab}\right)\eta_{(ab)} = \Lambda^*\,\eta_{(ab)}, \quad a \neq b. \tag{134}$$

Now we proceed to the analysis of the main equations Eq.(129,130) and (134) in various cases.

B.2 Stability of the replica symmetric solution.

The entries of the matrix Q in this case are given by $q_{ab} = q_0 + (q_d - q_0)\delta_{ab}$, so its inverse Q^{-1} has the same form $(Q^{-1})_{ab} = p_0 + (p_d - p_0)\delta_{ab}$, with p_0 and p_d defined in Eqs.(19,20). Taking this fact reduces the equation Eqs.(129) for $a \neq b$ to the form (see Eq.(132)):

$$p_0^2\sum_{(cd)}\eta_{(cd)} + p_0(p_d - p_0)\left[\sum_{d}\eta_{(ad)} + \sum_{d}\eta_{(bd)}\right] + (p_d - p_0)^2\,\eta_{(ab)} + \frac{1}{T^2}f''(D)\delta D_{ab} = \Lambda^*\,\eta_{(ab)} \tag{135}$$

where $D = q_d - q_0$. The equation Eq.(130) for $a = b$ is similarly reduced to

$$p_0^2 \sum_{(cd)} \eta_{(cd)} + 2p_0(p_d - p_0) \sum_d \eta_{(ad)} + (p_d - p_0)^2 \, \eta_{(aa)} - \frac{1}{T^2} f''(D) \sum_c \delta D_{ac} = \Lambda^* \, \eta_{(aa)}, \quad \forall \, a \tag{136}$$

Now we can follow faithfully the lines of the classical work by De Almeida and Thouless [36] and to provide an explicit construction of the eigenvectors with components $\eta_{(ab)}$.

The first ("longitudinal") family consists of those eigenvectors which have the same form as the replica-symmetric Q matrices themselves: $\eta^{(I)}_{(ab)} = \beta + (\alpha - \beta)\delta_{ab}$. Then $\sum_d \eta^{(I)}_{(ad)} = \alpha + \beta\,(n-1)$, and Eqs.(135,136) is reduced in the replica limit $n \to 0$ to a system of two equations

$$2p_0(p_d - p_0)(\alpha - \beta) + (p_d - p_0)^2 \, \beta + \frac{1}{T^2} f''(D)(\alpha - \beta) = \Lambda^* \, \beta \tag{137}$$

$$2p_0(p_d - p_0)(\alpha - \beta) + (p_d - p_0)^2 \, \alpha + \frac{1}{T^2} f''(D)(\alpha - \beta) = \Lambda^* \, \alpha \tag{138}$$

One can find the corresponding two eigenvalues easily, but for our goals it is enough to notice that for any values of the system parameters the eigenvalue Λ^* can not vanish as $\Lambda^* = 0$ immediately implies $\beta = \alpha = 0$. Hence, fluctuations in the longitudinal family cannot induce instability of the replica-symmetric solution.

Second family $\eta^{(II)}$ of the eigenvectors can be constructed as those characterized by one particular replica index singled out among all n. As all these choices are equivalent up to permutation of replica indices, we can take for definiteness $a = 1$. At this special choice of the replica index we have explicitly

$$\eta^{(II)}_{(11)} = \omega, \; \eta^{(II)}_{(1b)} = \eta^{(II)}_{(b1)} = \tau, \, \forall \, b \neq 1, \text{ and} \, \eta^{II}_{(ab)} = \mu + \delta_{ab}(\nu - \mu), \, \forall \, a \neq 1 \, ; \forall \, b \neq 1, \tag{139}$$

To ensure orthogonality of such an eigenvector to those of the longitudinal family, i.e. $(\eta^{(II)}, \eta^{(I)}) = 0$, see Eq.(120), amounts to imposing the conditions

$$\omega + (n-1)\nu = 0, \quad \tau + \mu \, \frac{n-2}{2} = 0, \tag{140}$$

It is easy to check that these conditions in fact ensure that

$$\sum_d \eta^{(II)}_{(1d)} = -(n-1)(\nu - \tau), \; \sum_d \eta^{(II)}_{(cd)} = \nu - \tau, \, \forall \, c \neq 1, \quad \text{so that} \quad \sum_{(cd)} \eta^{(II)}_{(cd)} = 0. \tag{141}$$

Note that in the replica limit $n \to 0$ the relations Eq.(140) imply $\omega = \nu$ and $\tau = \mu$. Then it is easy to check that the equations Eq.(135,136) for $a = 1$ are reduced to precisely the same system of two equations as that in Eq.(137,138), with the correspondence $\alpha \to \nu, \quad \beta \to \tau$. Moreover, for any $a > 1$ the equations Eq.(135,136) yield once more the same pair of equations. We therefore conclude that the fluctuations corresponding to this

second family of eigenvectors are also unable to induce instability of the replica-symmetric solution.

Finally, the third family of eigenvectors $\eta^{(III)}$ orthogonal to the first two turns out to be characterised by a chosen pair of nonequal replica indices (a,b), $a \neq b$. For the sake of definiteness we can take $a=1, b=2$ when the corresponding eigenvector components are given explicitly by:

$$\eta^{(III)}_{(aa)} = 0,\, \forall\, a;\;\; \eta^{(III)}_{(12)} = \eta^{(III)}_{(21)} = \xi, \tag{142}$$

$$\eta^{(III)}_{(1b)} = \eta^{(III)}_{(2b)} = \eta^{(III)}_{(b1)} = \eta^{(III)}_{(b2)} = \psi,\;\; \forall\, b > 2,$$

$$\eta^{III}_{(ab)} = (1-\delta_{ab})\rho,\, \forall\, a > 2\,; \forall\, b > 2$$

where the parameters ξ, ψ, ρ satisfy the constraints:

$$\xi + (n-2)\psi = 0, \quad 2\psi + \rho\,(n-3) = 0, \tag{143}$$

It is easy to check that these conditions in fact ensure that

$$\sum_d \eta^{(III)}_{(ad)} = 0, \forall\, a \tag{144}$$

so that automatically $\sum_{(cd)} \eta^{(III)}_{(cd)} = 0$. We then find that the equations Eq.(136) are satisfied identically, whereas the equations Eq.(135) are reduced to the form

$$\left[(p_d - p_0)^2 - \frac{1}{T^2} f''(D)\right] \eta^{(III)}_{(ab)} = \Lambda^* \eta^{(III)}_{(ab)}, \quad \forall\, a \neq b. \tag{145}$$

As the replica symmetric solution in the $\mu-$dominated regime implies $D = T/\mu$, $p_d - p_0 = \mu/T$, see Eqs.(21, 24), we infer from Eq.(145) that the stability condition $\Lambda^* \geq 0$ indeed implies the inequality Eq.(31).

Turning to the stability of $R-$dominated replica-symmetric solution, we find that the eigen-equations Eq.(134) are reduced to

$$p_0^2 \sum_{(cd)} \eta_{(cd)} + p_0(p_d - p_0)\left[\sum_d \eta_{(ad)} + \sum_d \eta_{(bd)}\right] + (p_d - p_0)^2\, \eta_{(ab)} \tag{146}$$

$$-\frac{1}{T^2} f''\left(R^2 - q_0\right) \eta_{(ab)} = \Lambda^*\, \eta_{(ab)}, \quad a \neq b$$

The subsequent analysis is completely analogous to one performed above in the $\mu-$dominated regime. In particular, the relevant eigenvalue again corresponds to the third family of eigenvectors satisfying the constraint Eq.(144). This fact, together with the relation $p_d - p_0 = 1/(R^2 - q_0)$ immediately yields the inequality Eq.(32) as the corresponding stability condition.

B.3 Stability of the Parisi solution with one step of the replica symmetry breaking.

In the following we will show that in this situation there are essentially two relevant eigenvalues, both in the $\mu-$dominated and $R-$dominated regimes:

$$\Lambda_K^* = \frac{1}{(q_d - q_1)^2} - \frac{1}{T^2} f''(q_d - q_1) \tag{147}$$

and

$$\Lambda_0^* = \frac{1}{(q_d - q_1 + m(q_1 - q_0))^2} - \frac{1}{T^2} f''(q_d - q_0) \ . \tag{148}$$

If both are positive, all other eigenvalues are positive and the system is stable. In the replica symmetric case both eigenvalues fall together. Let us in general note that if we know the solution on the line $T = T_b(\mu)$, which is defined by the condition $q_d = R^2$, we know the solution in the whole R-dominated regime, since there the values q_d , q_1 , q_0 and m are independent of μ (see the main text of the paper). Thus it is in general enough to proof the stability in the μ-dominated regime.

To this end, let us consider the function $R(q) = S(q)/T$, where $S(q)$ was defined in Eq.(49) of the main text. Specified for the case of 1RSB this function satisfies (cf.(69)):

$$R(q_0) = 0, \qquad R(q_1) = 0, \qquad \text{and} \qquad \int_{q_0}^{q_1} R(q)\, dq = 0 \ . \tag{149}$$

Moreover we know that $R(q)$ must change sign once between q_0 and q_1. Let us calculate the positions of the corresponding two extrema $q_e^{(1,2)}$ from $R'(q_e) = 0$:

$$\frac{1}{(q_d - q_1 + m(q_1 - q_e))^2} - \frac{1}{T^2} f''(q_d - q_e) = 0 \ . \tag{150}$$

which yields

$$q_d - q_1 + m(q_1 - q_e) = \frac{T}{\sqrt{f''(q_d - q_e)}} \tag{151}$$

for $q_e = q_e^{(1,2)}$. Since the right-hand side is convex according to our assumption for short range correlations, we have $q_d - q_1 + m(q_1 - q) > T/\sqrt{f''(q_d - q)}$ for $q_e^{(1)} < q < q_e^{(2)}$, hence $R'(q) < 0$ in that interval. This however means that for $q \notin (q_e^{(1)}, q_e^{(2)})$ necesserily $R'(q) > 0$, in particular $R'(q_0) > 0$ and $R'(q_1) > 0$. (The accidental case $R'(q_0) = 0$ or $R'(q_1) = 0$ happens on the AT-line). Note however that $R'(q_0)$ is exactly equal to Λ_0^* and similarly $R'(q_1) = \Lambda_K^*$. This observation proves stability everywhere in the $\mu-$ dominated phase. The stability condition becomes marginal along the AT-line.

In the rest of this appendix we will sketch the derivation of the fluctuation eigenvalues in both regimes, μ-dominated and R-dominated. First let us define a Parisi-block matrix

$P_{ab} = 1$ inside a diagonal Parisi $m \times m$ block and zero outside. With this definition the entries of the Parisi matrix Q has the form

$$q_{ab} = q_0 + (q_1 - q_0)P_{ab} + (q_d - q_1)\delta_{ab} \; . \tag{152}$$

The inverse matrix Q^{-1} has the same form:

$$\left(Q^{-1}\right)_{ab} = A_0 + (A_1 - A_0)P_{ab} + (A_d - A_1)\delta_{ab} \; . \tag{153}$$

The matrix P_{ab} can be used to introduce the following shorthand notations for several types of averages of the eigenvector components $\eta_{(ab)}$ over the indices inside a Parisi block:

$$\eta_{a\bar{b}} = \frac{1}{m}(\eta\ P)_{ab}, \quad \eta_{\bar{a}\bar{b}} = \frac{1}{m^2}(P\eta P)_{ab}, \quad \eta_{\overline{aa}} = \frac{1}{m}\sum_c \eta_{(cc)}\ P_{ca} \; . \tag{154}$$

With these definitions the eigen-equations Eq.(131) can be written as

$$\begin{aligned}
\Lambda^*\eta_{ab} \;\; = \;\; & A_0^2 \sum_{c,d} \eta_{(cd)} + (A_1 - A_0)^2 m^2 \eta_{\bar{a}\bar{b}} + (A_d - A_1)^2 \eta_{(ab)} \\
& + m A_0 (A_1 - A_0) \sum_c (\eta_{\bar{a}c} + \eta_{c\bar{b}}) + A_0 (A_d - A_1) \sum_c (\eta_{(ac)} + \eta_{(cb)}) \\
& + (A_1 - A_0)(A_d - A_1) m (\eta_{\bar{a}b} + \eta_{a\bar{b}}) \\
& + \frac{1}{T^2} f''(D_{ab}) \delta D_{ab} - \delta_{ab} \sum_c \frac{1}{T^2} f''(D_{ac}) \delta D_{ac}.
\end{aligned} \tag{155}$$

Here, explicitly

$$f''(D_{ab}) = f''(q_d - q_0) + (f''(q_d - q_1) - f''(q_d - q_0))P_{ab} \tag{156}$$

and

$$\begin{aligned}
2\sum_c f''(D_{ac})\delta D_{ac} = f''(q_d - q_0) \left(n\, \eta_{(aa)} + \sum_c \eta_{(cc)} - 2\sum_c \eta_{(ac)} \right) + \\
+ \quad m \left[f''(q_d - q_1) - f''(q_d - q_0) \right] \left(\eta_{(aa)} + \eta_{\overline{aa}} - 2\eta_{a\bar{a}} \right) .
\end{aligned} \tag{157}$$

Now one observes that one can derive a closed system of three equations for the following three combinations

$$\sum_a \eta_{(aa)}, \qquad \sum_a \eta_{\bar{a}\bar{a}}, \qquad \sum_{ab} \eta_{(ab)} \; .$$

This yields three eigenvalues in the limit $n \to 0$:

$$\begin{aligned}
\Lambda_1^* \;\; &= \;\; (A_d - A_1 + m(A_1 - A_0))^2 \\
\Lambda_2^* \;\; &= \;\; (A_d - A_1)^2 - \frac{1}{T^2} f''(q_d - q_1) + \frac{1-m}{T^2}(f''(q_d - q_1) - f''(q_d - q_0)) \\
\Lambda_3^* \;\; &= \;\; (A_d - A_1 + m(A_1 - A_0))^2 - \frac{1}{T^2} f''(q_d - q_0) \; .
\end{aligned} \tag{158}$$

As the next family of eigenvectors we use those satisfying the constraint $\sum_a \eta_{(aa)} = \sum_a \eta_{\bar{a}\bar{a}} = \sum_{ab} \eta_{(ab)} \;\; = 0$. One can show that constraints of this sort ensure orthogonality of the new families of eigenvectors to the old one (compare with the de-Almeida-Thouless analysis of the previous section). Subsequently, one derives a system of equations for

$$\eta_{\overline{aa}}, \qquad \eta_{\bar{a}\bar{a}}, \qquad \sum_b \eta_{\bar{a}b} \; .$$

In the limit $n \to 0$ that system yields the same eigenvalues $\Lambda_1^*, \Lambda_2^*, \Lambda_3^*$.

Using now as the constraint the conditions $\eta_{\overline{aa}} = \eta_{\bar{a}\bar{a}} = \sum_b \eta_{\bar{a}b} \;\; = 0$ simultaneously for all a we derive a single equation for the component $\eta_{\bar{a}\bar{b}}$, with the indices a and b in different Parisi blocks (we write $\bar{a} \neq \bar{b}$). The resulting eigenvalue is again Λ_3^*.

In the next step we impose another constraint $\eta_{\bar{a}\bar{b}} = 0$ for all a, b to obtain a set of equations for

$$\eta_{(aa)}, \qquad \eta_{a\bar{a}}, \qquad \sum_b \eta_{(ab)} \; .$$

This leads to another three eigenvalues

$$\begin{aligned}
\Lambda_4^* &= (A_d - A_1)(A_d - A_1 + m(A_1 - A_0)) \\
\Lambda_5^* &= (A_d - A_1)^2 - \frac{1}{T^2} f''(q_d - q_1) + \frac{2-m}{2T^2}(f''(q_d - q_1) - f''(q_d - q_0)) \\
\Lambda_6^* &= (A_d - A_1)(A_d - A_1 + m(A_1 - A_0)) - \frac{1}{T^2} f''(q_d - q_0) \; . \qquad (159)
\end{aligned}$$

We then obtain a single equation for $\eta_{a\bar{b}}$, which leads for $\bar{a} \neq \bar{b}$ to Λ_6^* again. The remaining equation for $\eta_{(ab)}$ yields for $\bar{a} \neq \bar{b}$ a new eigenvalue

$$\Lambda_7^* = (A_d - A_1)^2 - \frac{1}{T^2} f''(q_d - q_0) \qquad (160)$$

and for $\bar{a} = \bar{b}$ and $a \neq b$

$$\Lambda_8^* = (A_d - A_1)^2 - \frac{1}{T^2} f''(q_d - q_1) \; . \qquad (161)$$

Assuming $f''(x)$ monotonously decreasing, $0 < m < 1$ and $0 < q_0 < q_1 < q_d$ the relevant eigenvalues are obviously $\Lambda_2^* = \Lambda_0^*$ and $\Lambda_8^* = \Lambda_K^*$. Here we use that the eigenvalues of the matrix Q are

$$q_d - q_1, \quad q_d - q_1 + m(q_1 - q_0), \quad q_d - q_1 + m(q_1 - q_0) + nq_0 \qquad (162)$$

These are inverses of the eigenvalues of the matrix Q^{-1}:

$$A_d - A_1, \quad A_d - A_1 + m(A_1 - A_0), \quad A_d - A_1 + m(A_1 - A_0) + nA_0 \; . \qquad (163)$$

Turning now to the R-dominated regime, the eigenvalue equations (134) with the use of the short-hand notations introduced in Eq.(132) can be written in the form:

$$\begin{aligned}
\Lambda^* \, \delta D_{ab} &= (A_0^2 \sum_{c,d} \delta D_{c,d} + (A_1 - A_0)^2 m^2 \delta D_{\bar a \bar b} + (A_d - A_1)^2 \delta D_{ab} \\
&+ mA_0(A_1 - A_0) \sum_c (\delta D_{\bar a c} + \delta Q_{c\bar b}) + A_0(A_d - A_1) \sum_c (\delta D_{ac} + \delta D_{cb}) \\
&+ (A_1 - A_0)(A_d - A_1) m(\delta D_{\bar a b} + \delta D_{a\bar b}))(1 - \delta_{ab}) - \frac{1}{T^2} f''(D_{ab}) \delta D_{ab}
\end{aligned}$$

We proceed in the same way as before. The equations for

$$\sum_a \delta D_{\bar a \bar a}, \qquad \sum_{ab} \delta D_{ab}$$

produce the quadratic equation $\begin{vmatrix} A - \Lambda^* & B \\ C & D - \Lambda^* \end{vmatrix} = 0$ with

$$\begin{aligned}
A &= (m-1)(m(A_1 - A_0)^2 + 2(A_1 - A_0)(A_d - A_1)) + (A_d - A_1)^2 - \frac{1}{T^2} f''(q_d - q_1) \\
B &= (m-1)(nA_0^2 + 2A_0(A_d - A_1 + m(A_1 - A_0))) \\
C &= -m(A_1 - A_0)^2 - 2(A_1 - A_0)(A_d - A_1) - \frac{1}{T^2} f''(q_d - q_1) + \frac{1}{T^2} f''(q_d - q_0) \\
D &= n(n-1)A_0^2 + 2(n-1)A_0(A_d - A_1 + m(A_1 - A_0)) + (A_d - A_1 + m(A_1 - A_0))^2 \\
&\quad - \frac{1}{T^2} f''(q_d - q_0) \, .
\end{aligned}$$

In the next step we obtain for

$$\delta D_{\bar a \bar a}, \qquad \sum_b \delta D_{\bar a b}$$

another quadratic equation $\begin{vmatrix} A - \Lambda^* & B \\ C & D - \Lambda^* \end{vmatrix} = 0$ with

$$\begin{aligned}
A &= (m-1)(m(A_1 - A_0)^2 + 2(A_1 - A_0)(A_d - A_1)) \ + \ (A_d - A_1)^2 - \frac{1}{T^2} f''(q_d - q_1) \\
B &= (m-1)(2A_0(A_d - A_1 + m(A_1 - A_0))) \\
C &= -m(A_1 - A_0)^2 - 2(A_1 - A_0)(A_d - A_1) - \frac{1}{T^2} f''(q_d - q_1) + \frac{1}{T^2} f''(q_d - q_0) \\
D &= (n-2)A_0(A_d - A_1 + m(A_1 - A_0)) + (A_d - A_1 + m(A_1 - A_0))^2 - \frac{1}{T^2} f''(q_d - q_0)
\end{aligned}$$

Both equations fall together for $n \to 0$ and produce the same eigenvalues Λ_1^*, Λ_2^*. One can show under the conditions we have that both eigenvalues are positive provided Λ_0^* and Λ_K^* are. Next one derives a single equation for $\delta D_{\bar a \bar b}$ for $\bar a \neq \bar b$ leading to the eigenvalue

$$\Lambda_3^* = (A_d - A_1 + m(A_1 - A_0))^2 - \frac{1}{T^2} f''(q_d - q_0) \tag{164}$$

Note that Λ_3^* here is formally equal to Λ_3^* found in the previous analysis of the $\mu-$dominated regime. On the next lower level we obtain for

$$\delta D_{aa}, \qquad \sum_b \delta D_{ab}$$

the eigenvalue equation $\begin{vmatrix} A-\Lambda^* & B \\ C & D-\Lambda^* \end{vmatrix} = 0$ with

$$\begin{aligned}
A &= (m-2)(A_1-A_0)(A_d-A_1)) \ + \ (A_d-A_1)^2 - \frac{1}{T^2} f''(q_d-q_1) \\
B &= (m-2)A_0(A_d-A_K) \\
C &= -2(A_1-A_0)(A_d-A_1) - \frac{1}{T^2} f''(q_d-q_1) + \frac{1}{T^2} f''(q_d-q_0) \\
D &= (A_d-A_1)((n-2)A_0+m(A_1-A_0)) + (A_d-A_1)^2 - \frac{1}{T^2} f''(q_d-q_0)
\end{aligned}$$

leading to Λ_4^*, Λ_5^*. Again both are positive if Λ_0^* and Λ_K^* are. On the next lower levels we again reproduce the previous eigenvalues Λ_6^*, Λ_7^* and Λ_8^* of the $\mu-$dominated analysis. As the result in both regimes are Λ_0^* and Λ_K^* the relevant eigenvalues, which have to be positive for stability - and they are indeed positive as we have demonstrated before.

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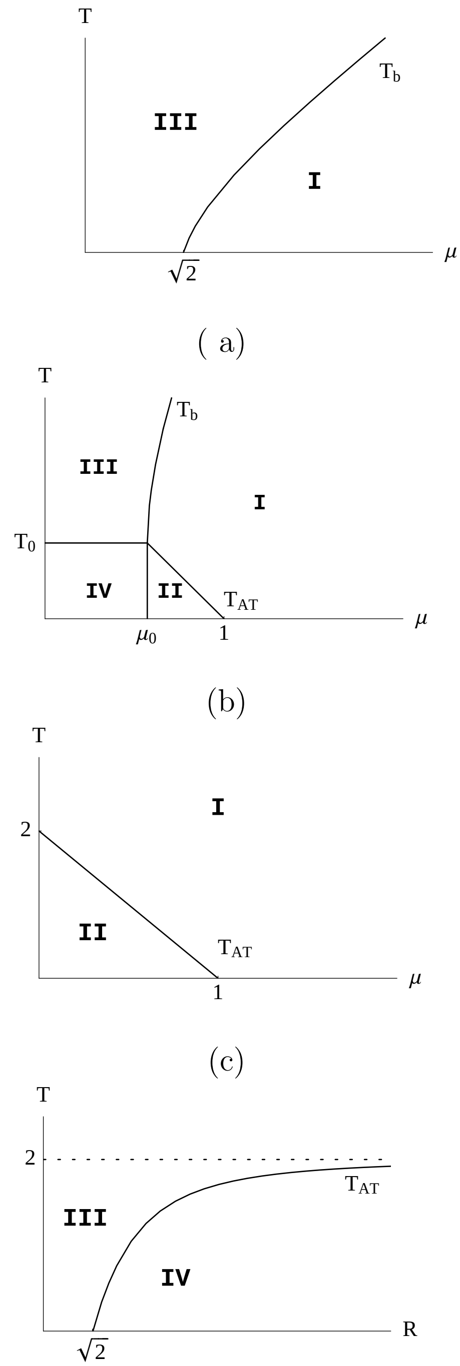
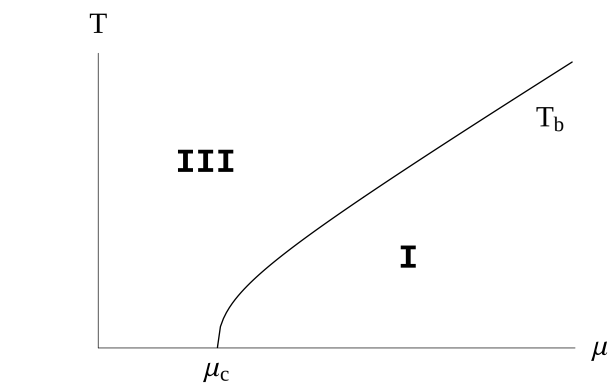
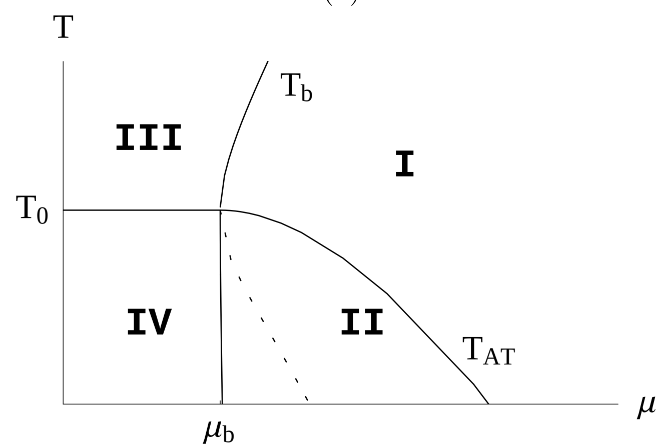


Figure 2: The phase diagrams for the potential with logarithmic correlations, with $g = 2, a = \sqrt{2}$. In the case (a) the radius of the box is $R = 1 < R_{cr} = \sqrt{2}$, in the case (b) $R = \sqrt{5} > R_{cr}$, in the case (c) $R = \infty$. The case (d) corresponds to the choice of parameter $\mu = 0$. The correct boundary T_b between the $R-$ dominated and $\mu-$ dominated glassy phases for the case (b) is represented by full vertical line. The notation for phases are the same as in fig.1.

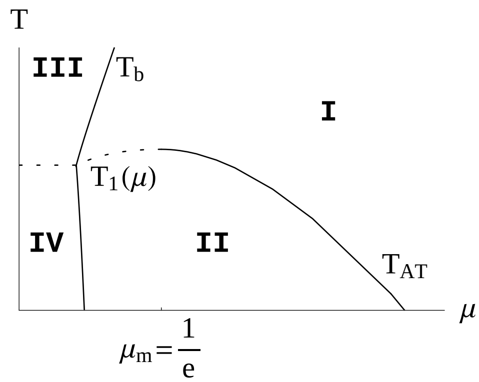


(a)

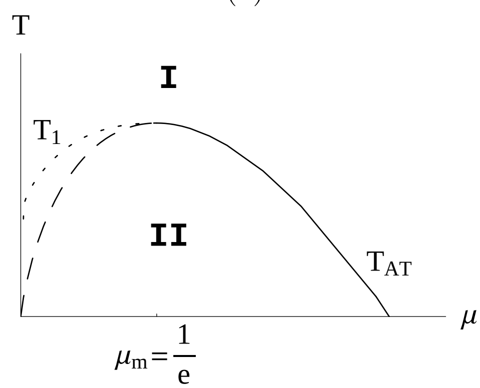


(b)

Figure 3: The phase diagrams for the potential with short-range correlations, $f(x) = e^{-x}$. In the case (a) the radius of the box is $R = 1 = R_{cr}$, in the case (b) $R = \sqrt{3} = R_*$. Dotted line represents the wrong branch of the boundary T_b between the $R-$ dominated and $\mu-$ dominated glassy phases and should be replaced by the full line close to vertical. The notation for phases are the same as in fig. 1.



(a)



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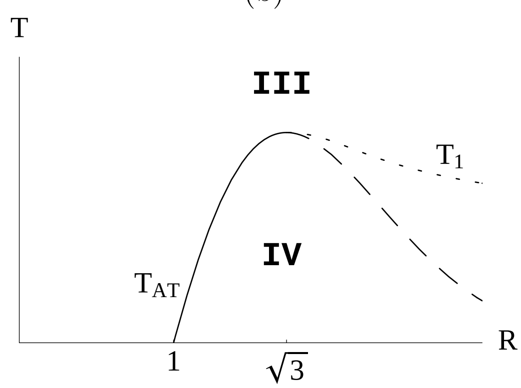

(c)

Figure 4: The phase diagrams for the potential with short-range correlations, $f(x) = e^{-x}$. In the case (a) the radius of the box is $R = \sqrt{5} > R_*$, in the case (b) $R = \infty$ and the case (c) corresponds to $\mu = 0$. The dotted curves in (a)-(c) represents the transition line $T_1(\mu)$ found from the condition $m \to 1$ and the broken curve in (B) and (c) represents the part of T_{AT} line below T_1. The notation for phases are the same as in fig. 1.